

# The Laplace Transformation

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$$L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx} f(x) \, dx$$

All the differential equations you have looked at so far have had solutions containing a number of unknown integration constants  $A, B, C$  etc. The values of these constants have then been found by applying boundary conditions to the solution, a procedure that can often prove to be tedious. Fortunately, for a certain type of differential equation there is a method of obtaining the solution where these unknown integration constants are evaluated *during the process of solution*. Furthermore, rather than employing integration as the way of unravelling the differential equation, you use straightforward algebra.

The method hinges on what is called the *Laplace transform*. If  $f(x)$  represents some expression in  $x$  defined for  $x \geq 0$ , the *Laplace transform* of  $f(x)$ , denoted by  $L\{f(x)\}$ , is defined to be:

$$L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx} f(x) dx$$

where  $s$  is a variable whose values are chosen so as to ensure that the semi-infinite integral converges. More will be said about the variable  $s$  in Frame 3. For now, what would you say is the Laplace transform  $f(x) = 2$  for  $x \geq 0$ ?

*Substitute for  $f(x)$  in the integral above and then perform the integration.  
The answer is in the next frame*

$$L\{2\} = \frac{2}{s} \text{ provided } s > 0$$

Because:

$$L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx} f(x) dx$$

so

$$\begin{aligned} L\{2\} &= \int_{x=0}^{\infty} e^{-sx} 2 dx \\ &= 2 \left[ \frac{e^{-sx}}{-s} \right]_{x=0}^{\infty} \\ &= 2(0 - (-1/s)) \\ &= \frac{2}{s} \end{aligned}$$

Notice that  $s > 0$  is demanded because if  $s < 0$  then  $e^{-sx} \rightarrow \infty$  as  $x \rightarrow \infty$  and if  $s = 0$  then  $L\{2\}$  is not defined (in both of these two cases the integral diverges), so that

$$L\{2\} = \frac{2}{s} \text{ provided } s > 0$$

By the same reasoning, if  $k$  is some constant then

$$L\{k\} = \frac{k}{s} \text{ provided } s > 0$$

Now, how about the Laplace transform of  $f(x) = e^{-kx}$ ,  $x \geq 0$  where  $k$  is a constant?

$$L\{e^{-kx}\} = \frac{1}{s + k} \text{ provided } s > -k$$

Because

$$\begin{aligned} L\{e^{-kx}\} &= \int_{x=0}^{\infty} e^{-sx} e^{-kx} dx \\ &= \int_{x=0}^{\infty} e^{-(s+k)x} dx \\ &= \left[ \frac{e^{-(s+k)x}}{-(s+k)} \right]_{x=0}^{\infty} \\ &= \left( 0 - \left( -\frac{1}{(s+k)} \right) \right) \quad s+k > 0 \text{ is demanded to ensure that the} \\ &\quad \text{integral converges at both limits} \\ &= \frac{1}{(s+k)} \quad \text{provided } s+k > 0, \text{ that is provided } s > -k \end{aligned}$$

These two examples have demonstrated that you need to be careful about the finite existence of the Laplace transform and not just take the integral definition without some thought. For the Laplace transform to exist the integrand

$$e^{-sx}f(x)$$

must converge to zero as  $x \rightarrow \infty$  and this will impose some conditions on the values of  $s$  for which the integral does converge and, hence, the Laplace transform exists. In this Programme you can be assured that there are no problems concerning the existence of any of the Laplace transforms that you will meet.

## The inverse Laplace transform

The Laplace transform is an expression in the variable  $s$  which is denoted by  $F(s)$ . It is said that  $f(x)$  and  $F(s) = L\{f(x)\}$  form a *transform pair*. This means that if  $F(s)$  is the *Laplace transform* of  $f(x)$  then  $f(x)$  is the *inverse Laplace transform* of  $F(s)$ . We write:

$$f(x) = L^{-1}\{F(s)\}$$

There is no simple integral definition of the inverse transform so you have to find it by working backwards. For example:

$$\text{if } f(x) = 4 \text{ then the Laplace transform } L\{f(x)\} = F(s) = \frac{4}{s}$$

so

$$\text{if } F(s) = \frac{4}{s} \text{ then the inverse Laplace transform } L^{-1}\{F(s)\} = f(x) = 4$$

It is this ability to find the Laplace transform of an expression and then reverse it that makes the Laplace transform so useful in the solution of differential equations, as you will soon see.

For now, what is the inverse Laplace transform of  $F(s) = \frac{1}{s-1}$ ?



$$L^{-1}\{F(s)\} = f(x) = e^x$$

Because you know that:

$$L\{e^{-kx}\} = \frac{1}{s+k} \text{ you can say that } L^{-1}\left\{\frac{1}{s+k}\right\} = e^{-kx}$$

$$\text{so when } k = -1, L^{-1}\left\{\frac{1}{s-1}\right\} = e^{-(-1)x} = e^x$$

To assist in the process of finding Laplace transforms and their inverses a table is used. In the next frame is a short table containing what you know to date.

## Table of Laplace transforms

$f(x) = L^{-1}\{F(s)\}$	$F(s) = L\{f(x)\}$
$k$	$\frac{k}{s} \quad s > 0$
$e^{-kx}$	$\frac{1}{s+k} \quad s > -k$

Reading the table from left to right gives the Laplace transform and reading the table from right to left gives the inverse Laplace transform.

## Revision summary

- 1** The *Laplace transform* of  $f(x)$ , denoted by  $L\{f(x)\}$ , is defined to be:

$$L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx} f(x) \, dx$$

where  $s$  is a variable whose values are chosen so as to ensure that the semi-infinite integral converges.

- 2** If  $F(s)$  is the *Laplace transform* of  $f(x)$  then  $f(x)$  is the *inverse Laplace transform* of  $F(s)$ . We write:

$$f(x) = L^{-1}\{F(s)\}$$

There is no simple integral definition of the inverse transform so you have to find it by working backwards using a *Table of Laplace transforms*.

## Revision exercise

**1** Find the Laplace transform of each of the following. In each case  $f(x)$  is defined for  $x \geq 0$  :

(a)  $f(x) = -3$

(b)  $f(x) = e$

(c)  $f(x) = e^{2x}$

(d)  $f(x) = -5e^{-3x}$

(e)  $f(x) = 2e^{7x-2}$

**2** Find the inverse Laplace transform of each of the following:

(a)  $F(s) = -\frac{1}{s}$

(b)  $F(s) = \frac{1}{s-5}$

(c)  $F(s) = \frac{3}{s+2}$

(d)  $F(s) = -\frac{3}{4s}$

(e)  $F(s) = \frac{1}{2s-3}$

1 (a)  $f(x) = -3$

Because  $L\{k\} = \frac{k}{s}$  provided  $s > 0$ ,  $L\{-3\} = -\frac{3}{s}$  provided  $s > 0$

(b)  $f(x) = e$

Because  $L\{k\} = \frac{k}{s}$  provided  $s > 0$ ,  $L\{e\} = \frac{e}{s}$  provided  $s > 0$

(c)  $f(x) = e^{2x}$

Because  $L\{e^{-kx}\} = \frac{1}{s+k}$  provided  $s > -k$ ,  $L\{e^{2x}\} = \frac{1}{s-2}$  provided  $s > 2$

(d)  $f(x) = -5e^{-3x}$

$$L\{-5e^{-3x}\} = \int_{x=0}^{\infty} e^{-sx}(-5e^{-3x}) dx = -5 \int_{x=0}^{\infty} e^{-sx}e^{-3x} dx = -5L\{e^{-3x}\}$$

$$L\{-5e^{-3x}\} = -\frac{5}{s+3} \text{ provided } s > -3$$

(e)  $f(x) = 2e^{7x-2}$

$$L\{2e^{7x-2}\} = \int_{x=0}^{\infty} e^{-sx}(2e^{7x-2}) dx = 2e^{-2} \int_{x=0}^{\infty} e^{-sx}e^{7x} dx = 2e^{-2}L\{e^{7x}\}$$

$$L\{2e^{7x-2}\} = \frac{2e^{-2}}{s-7} \text{ provided } s > 7$$

2 (a)  $F(s) = -\frac{1}{s}$

Because  $L^{-1}\left\{\frac{k}{s}\right\} = k$ ,  $L^{-1}\left\{-\frac{1}{s}\right\} = L^{-1}\left\{\frac{-1}{s}\right\} = -1$

(b)  $F(s) = \frac{1}{s-5}$

Because  $L^{-1}\left\{\frac{1}{s+k}\right\} = e^{-kx}$ ,  $L^{-1}\left\{\frac{1}{s-5}\right\} = e^{-(-5)x} = e^{5x}$

(c)  $F(s) = \frac{3}{s+2}$

Because  $L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2x}$  and  $L\{3e^{-2x}\} = 3L\{e^{-2x}\} = \frac{3}{s+2}$  so

$$L^{-1}\left\{\frac{3}{s+2}\right\} = 3e^{-2x}$$

(d)  $F(s) = -\frac{3}{4s}$

$F(s) = -\frac{3}{4s} = \frac{(-3/4)}{s}$  so that  $L^{-1}\left\{-\frac{3}{4s}\right\} = L^{-1}\left\{\frac{-3/4}{s}\right\} = -3/4$

(e)  $F(s) = \frac{1}{2s-3}$

$F(s) = \frac{1}{2s-3} = \frac{\frac{1}{2}}{s-\frac{3}{2}}$  so that  $f(x) = L^{-1}\left\{\frac{1}{2s-3}\right\} = L^{-1}\left\{\frac{\frac{1}{2}}{s-\frac{3}{2}}\right\} = \frac{1}{2}e^{\frac{3}{2}x}$

## Laplace transform of a derivative

Before you can use the Laplace transform to solve a differential equation you need to know the Laplace transform of a derivative. Given some expression  $f(x)$  with Laplace transform  $L\{f(x)\} = F(s)$ , the Laplace transform of the derivative  $f'(x)$  is:

$$L\{f'(x)\} = \int_{x=0}^{\infty} e^{-sx} f'(x) dx$$

This can be integrated by parts as follows:

$$\begin{aligned} L\{f'(x)\} &= \int_{x=0}^{\infty} e^{-sx} f'(x) dx \\ &= \int_{x=0}^{\infty} u(x) dv(x) \\ &= \left[ u(x)v(x) \right]_{x=0}^{\infty} - \int_{x=0}^{\infty} v(x) du(x) \quad \text{(the Parts formula – see} \\ &\quad \text{Programme 15, Frame 21)} \end{aligned}$$

where  $u(x) = e^{-sx}$  so  $du(x) = -se^{-sx}dx$  and where  $dv(x) = f'(x)dx$  so  $v(x) = f(x)$ .

Therefore, substitution in the Parts formula gives:

$$\begin{aligned} L\{f'(x)\} &= \left[ e^{-sx} f(x) \right]_{x=0}^{\infty} + s \int_{x=0}^{\infty} e^{-sx} f(x) dx \\ &= (0 - f(0)) + sF(s) \text{ assuming } e^{-sx} f(x) \rightarrow 0 \text{ as } x \rightarrow \infty \end{aligned}$$

That is:

$$L\{f'(x)\} = sF(s) - f(0)$$

So the Laplace transform of the derivative of  $f(x)$  is given in terms of the Laplace transform of  $f(x)$  itself and the value of  $f(x)$  when  $x = 0$ . Before you use this fact just consider two properties of the Laplace transform in the next frame.



## Two properties of Laplace transforms

Both the Laplace transform and its inverse are *linear transforms*, by which is meant that:

- (1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is:*

$$L\{f(x) \pm g(x)\} = L\{f(x)\} \pm L\{g(x)\}$$

$$\text{and } L^{-1}\{F(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

- (2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is:*

$$L\{kf(x)\} = kL\{f(x)\} \text{ and } L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\} \text{ where } k \text{ is a constant}$$

These are easily proved using the basic definition of the Laplace transform in Frame 1.

Armed with this information let's try a simple differential equation. By using

$$L\{f'(x)\} = sF(s) - f(0)$$

take the Laplace transform of both sides of the equation

$$f'(x) + f(x) = 1 \text{ where } f(0) = 0$$

and find an expression for the Laplace transform  $F(s)$ .

$$F(s) = \frac{1}{s(s+1)}$$

Because, taking Laplace transforms of both sides of the equation you have that:

$$L\{f'(x) + f(x)\} = L\{1\}$$

The Laplace transform of the left-hand side equals the Laplace transform of the right-hand side

That is:

$$L\{f'(x)\} + L\{f(x)\} = L\{1\}$$

The transform of a sum is the sum of the transforms.

From what you know about the Laplace transform of  $f(x)$  and its derivative  $f'(x)$  this gives:

$$[sF(s) - f(0)] + F(s) = \frac{1}{s}$$

That is:

$$(s + 1)F(s) - f(0) = \frac{1}{s}$$

and you are given that  $f(0) = 0$  so

$$(s + 1)F(s) = \frac{1}{s}, \text{ that is } F(s) = \frac{1}{s(s + 1)}$$

Well done. Now, separate the right-hand side into partial fractions.

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

Because

Assume that  $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$  then,  $1 = A(s+1) + Bs$  from which you

find that  $A = 1$  and  $B = -1$  so that  $F(s) = \frac{1}{s} - \frac{1}{s+1}$

That was straightforward enough. Now take the inverse Laplace transform and find the solution to the differential equation.

$$f(x) = 1 - e^{-x}$$

Because

$$\begin{aligned} f(x) &= L^{-1}\{F(s)\} \\ &= L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\ &= L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} \quad \text{The inverse Laplace transform of a difference} \\ &\quad \text{is the difference of the inverse transforms} \\ &= 1 - e^{-x} \quad \text{Using the Table of Laplace transforms in Frame 6} \end{aligned}$$

You now have a method for solving a differential equation of the form:

$$af'(x) + bf(x) = g(x) \text{ given that } f(0) = k$$

where  $a$ ,  $b$  and  $k$  are known constants and  $g(x)$  is a known expression in  $x$ :

- (a) Take the Laplace transform of both sides of the differential equation
- (b) Find the expression  $F(s) = L\{f(x)\}$  in the form of an algebraic fraction
- (c) Separate  $F(s)$  into its partial fractions
- (d) Find the inverse Laplace transform  $L^{-1}\{F(s)\}$  to find the solution  $f(x)$  to the differential equation.

# Table of Laplace transforms

$f(x) = L^{-1}\{F(s)\}$	$F(s) = L\{f(x)\}$
$k$	$\frac{k}{s} \quad s > 0$
$e^{-kx}$	$\frac{1}{s+k} \quad s > -k$
$xe^{-kx}$	$\frac{1}{(s+k)^2} \quad s > -k$

## Revision summary

- 1** If  $F(s)$  is the Laplace transform of  $f(x)$  then the Laplace transform of  $f'(x)$  is:

$$L\{f'(x)\} = sF(s) - f(0)$$

- 2** (a) The Laplace transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is:

$$L\{f(x) \pm g(x)\} = L\{f(x)\} \pm L\{g(x)\}$$

$$\text{and } L^{-1}\{F(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

- (b) The transform of an expression multiplied by a constant is the constant multiplied by the transform of the expression. That is:

$$L\{kf(x)\} = kL\{f(x)\} \text{ and } L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$$

where  $k$  is a constant.

- 3** To solve a differential equation of the form:

$$af'(x) + bf(x) = g(x) \text{ given that } f(0) = k$$

where  $a$ ,  $b$  and  $k$  are known constants and  $g(x)$  is a known expression in  $x$ :

- Take the Laplace transform of both sides of the differential equation
- Find the expression  $F(s) = L\{f(x)\}$  in the form of an algebraic fraction
- Separate  $F(s)$  into its partial fractions
- Find the inverse Laplace transform  $L^{-1}\{F(s)\}$  to find the solution  $f(x)$  to the differential equation.

## Revision exercise

Solve each of the following differential equations:

(a)  $f'(x) - f(x) = 2$  where  $f(0) = 0$

(b)  $f'(x) + f(x) = e^{-x}$  where  $f(0) = 0$

(c)  $f'(x) + f(x) = 3$  where  $f(0) = -2$

(d)  $f'(x) - f(x) = e^{2x}$  where  $f(0) = 1$

(e)  $3f'(x) - 2f(x) = 4e^{-x} + 2$  where  $f(0) = 0$



(a)  $f'(x) - f(x) = 2$  where  $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$$sF(s) - f(0) - F(s) = \frac{2}{s} \text{ so that } F(s) = \frac{2}{s(s-1)} = -\frac{2}{s} + \frac{2}{s-1}$$

The inverse transform then gives the solution as

$$f(x) = -2 + 2e^x = 2(e^x - 1)$$

(b)  $f'(x) + f(x) = e^{-x}$  where  $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$$sF(s) - f(0) + F(s) = \frac{1}{s+1} \text{ so that } F(s) = \frac{1}{(s+1)^2}$$

The Table of inverse transforms then gives the solution as  $f(x) = xe^{-x}$

(c)  $f'(x) + f(x) = 3$  where  $f(0) = -2$

Taking Laplace transforms of both sides of this equation gives:

$$sF(s) - f(0) + F(s) = \frac{3}{s} \text{ so that}$$

$$F(s) = -\frac{2}{s+1} + \frac{3}{s(s+1)} = \frac{3-2s}{s(s+1)} = \frac{3}{s} - \frac{5}{s+1}$$

The inverse transform then gives the solution as  $f(x) = 3 - 5e^{-x}$

(d)  $f'(x) - f(x) = e^{2x}$  where  $f(0) = 1$

Taking Laplace transforms of both sides of this equation gives:

$$sF(s) - f(0) - F(s) = \frac{1}{s-2} \text{ giving } (s-1)F(s) - 1 = \frac{1}{s-2}$$

$$\text{so that } F(s) = \frac{1}{s-1} + \frac{1}{(s-1)(s-2)} = \frac{1}{s-2}$$

The inverse transform then gives the solution as  $f(x) = e^{2x}$

(e)  $3f'(x) - 2f(x) = 4e^{-x} + 2$  where  $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$$3[sF(s) - f(0)] - 2F(s) = \frac{4}{s+1} + \frac{2}{s} = \frac{6s+2}{s(s+1)} \text{ so that}$$

$$\begin{aligned} F(s) &= \frac{6s+2}{s(s+1)(3s-2)} = \frac{27}{5} \left( \frac{1}{3s-2} \right) - \frac{1}{s} - \frac{4}{5} \left( \frac{1}{s+1} \right) \\ &= \frac{27}{15} \left( \frac{1}{s - \frac{2}{3}} \right) - \frac{1}{s} - \frac{4}{5} \left( \frac{1}{s+1} \right) \end{aligned}$$

The inverse transform then gives the solution as:

$$f(x) = \frac{9}{5}e^{2x/3} - \frac{4}{5}e^{-x} - 1$$

## Generating new transforms

Deriving the Laplace transform of  $f(x)$  often requires you to integrate by parts, sometimes repeatedly. However, because  $L\{f'(x)\} = sL\{f(x)\} - f(0)$  you can sometimes avoid this involved process when you know the transform of the derivative  $f'(x)$ . Take as an example the problem of finding the Laplace transform of the expression  $f(x) = x$ . Now  $f'(x) = 1$  and  $f(0) = 0$  so that substituting in the equation:

$$L\{f'(x)\} = sL\{f(x)\} - f(0)$$

gives

$$L\{1\} = sL\{x\} - 0$$

that is

$$\frac{1}{s} = sL\{x\}$$

therefore

$$L\{x\} = \frac{1}{s^2}$$

That was easy enough, so what is the Laplace transform of  $f(x) = x^2$ ?

$$\frac{2}{s^3}$$

Because

$$f(x) = x^2, \quad f'(x) = 2x \text{ and } f(0) = 0$$

Substituting in

$$L\{f'(x)\} = sL\{f(x)\} - f(0)$$

gives

$$L\{2x\} = sL\{x^2\} - 0$$

that is

$$2L\{x\} = sL\{x^2\} \text{ so } \frac{2}{s^2} = sL\{x^2\}$$

therefore

$$L\{x^2\} = \frac{2}{s^3}$$

Just try another one. Verify the third entry in the Table of Laplace transforms in Frame 15, that is:

$$L\{xe^{-x}\} = \frac{1}{(s+1)^2}$$

Because

$$f(x) = xe^{-x}, \quad f'(x) = e^{-x} - xe^{-x} \text{ and } f(0) = 0$$

Substituting in

$$L\{f'(x)\} = sL\{f(x)\} - f(0)$$

gives

$$L\{e^{-x} - xe^{-x}\} = sL\{xe^{-x}\} - 0$$

that is

$$L\{e^{-x}\} - L\{xe^{-x}\} = sL\{xe^{-x}\}$$

therefore

$$L\{e^{-x}\} = (s + 1)L\{xe^{-x}\}$$

giving

$$\frac{1}{s + 1} = (s + 1)L\{xe^{-x}\} \text{ and so } L\{xe^{-x}\} = \frac{1}{(s + 1)^2}$$

## Laplace transforms of higher derivatives

The Laplace transforms of derivatives higher than the first are readily derived. Let  $F(s)$  and  $G(s)$  be the respective Laplace transforms of  $f(x)$  and  $g(x)$ . That is

$$L\{f(x)\} = F(s) \text{ so that } L\{f'(x)\} = sF(s) - f(0)$$

and

$$L\{g(x)\} = G(s) \text{ and } L\{g'(x)\} = sG(s) - g(0)$$

Now let  $g(x) = f'(x)$  so that  $L\{g(x)\} = L\{f'(x)\}$  where

$$g(0) = f'(0) \text{ and } G(s) = sF(s) - f(0)$$

Now, because  $g(x) = f'(x)$

$$g'(x) = f''(x)$$

This means that

$$L\{g'(x)\} = L\{f''(x)\} = sG(s) - g(0) = s[sF(s) - f(0)] - f'(0)$$

so

$$L\{f''(x)\} = s^2F(s) - sf(0) - f'(0)$$

By a similar argument it can be shown that

$$L\{f'''(x)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

and so on. Can you see the pattern developing here?

The Laplace transform of  $f^{iv}(x)$  is .....

$$L\{f^{iv}(x)\} = s^4 F(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$



Now, using  $L\{f''(x)\} = s^2F(s) - sf(0) - f'(0)$  the Laplace transform of  $f(x) = \sin kx$  where  $k$  is a constant is .....

$$L\{\sin kx\} = \frac{k}{s^2 + k^2}$$

Because

$$f(x) = \sin kx, f'(x) = k \cos kx \text{ and } f''(x) = -k^2 \sin kx.$$

$$\text{Also } f(0) = 0 \text{ and } f'(0) = k.$$

Substituting in

$$L\{f''(x)\} = s^2F(s) - sf(0) - f'(0) \text{ where } F(s) = L\{f(x)\} \\ \text{gives}$$

$$L\{-k^2 \sin kx\} = s^2L\{\sin kx\} - s.0 - k$$

that is

$$-k^2L\{\sin kx\} = s^2L\{\sin kx\} - k$$

so

$$(s^2 + k^2)L\{\sin kx\} = k \text{ and } L\{\sin kx\} = \frac{k}{s^2 + k^2}$$

$$\text{And } L\{\cos kx\} = \dots\dots\dots$$

$$L\{\cos kx\} = \frac{s}{s^2 + k^2}$$

Because

$$f(x) = \cos kx, f'(x) = -k \sin kx \text{ and } f''(x) = -k^2 \cos kx.$$

$$\text{Also } f(0) = 1 \text{ and } f'(0) = 0.$$

Substituting in

$$L\{f''(x)\} = s^2 F(s) - sf(0) - f'(0) \text{ where } F(s) = L\{f(x)\}$$

gives

$$L\{-k^2 \cos kx\} = s^2 L\{\cos kx\} - s.1 - 0$$

that is

$$-k^2 L\{\cos kx\} = s^2 L\{\cos kx\} - s$$

so

$$(s^2 + k^2)L\{\cos kx\} = s \text{ and } L\{\cos kx\} = \frac{s}{s^2 + k^2}$$

The Table of transforms is now extended in the next frame.

# Table of Laplace transforms

$f(x) = L^{-1}\{F(s)\}$	$F(s) = L\{f(x)\}$
$k$	$\frac{k}{s} \quad s > 0$
$e^{-kx}$	$\frac{1}{s+k} \quad s > -k$
$xe^{-kx}$	$\frac{1}{(s+k)^2} \quad s > -k$
$x$	$\frac{1}{s^2} \quad s > 0$
$x^2$	$\frac{2}{s^3} \quad s > 0$
$\sin kx$	$\frac{k}{s^2 + k^2} \quad s^2 + k^2 > 0$
$\cos kx$	$\frac{s}{s^2 + k^2} \quad s^2 + k^2 > 0$

## Linear, constant-coefficient, inhomogeneous differential equations

The Laplace transform can be used to solve equations of the form:

$$a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_2 f''(x) + a_1 f'(x) + a_0 f(x) = g(x)$$

where  $a_n, a_{n-1}, \dots, a_2, a_1, a_0$  are known constants,  $g(x)$  is a known expression in  $x$  and the values of  $f(x)$  and its derivatives are known at  $x = 0$ . This type of equation is called a *linear, constant-coefficient, inhomogeneous differential equation* and the values of  $f(x)$  and its derivatives at  $x = 0$  are called *boundary conditions*. The method of obtaining the solution follows the procedure laid down in Frame 14. For example:

To find the solution of:

$$f''(x) + 3f'(x) + 2f(x) = 4x \text{ where } f(0) = f'(0) = 0$$

(a) *Take the Laplace transform of both sides of the equation*

$$L\{f''(x)\} + 3L\{f'(x)\} + 2L\{f(x)\} = 4L\{x\}$$

$$\text{to give } [s^2 F(s) - sf(0) - f'(0)] + 3[sF(s) - f(0)] + 2F(s) = \frac{4}{s^2}$$

(b) Find the expression  $F(s) = L\{f(x)\}$  in the form of an algebraic fraction

Substituting the values for  $f(0)$  and  $f'(0)$  and then rearranging gives

$$(s^2 + 3s + 2)F(s) = \frac{4}{s^2}$$

so that

$$F(s) = \frac{4}{s^2(s+1)(s+2)}$$

(c) Separate  $F(s)$  into its partial fractions

$$\frac{4}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

Adding the right-hand side partial fractions together and then equating the left-hand side numerator with the right-hand side numerator gives

$$4 = As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$$

$$\text{Let } s = 0 \quad 4 = 2B \text{ therefore } B = 2$$

$$s = -1 \quad 4 = C(-1)^2(-1+2) = C$$

$$s = -2 \quad 4 = D(-2)^2(-2+1) = -4D \text{ therefore } D = -1$$

Equate the coefficients of  $s$ :

$$0 = 2A + 3B = 2A + 6 \text{ therefore } A = -3$$

Consequently:

$$F(s) = -\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2}$$

(d) *Use the Tables to find the inverse Laplace transform  $L^{-1}\{F(s)\}$  and so find the solution  $f(x)$  to the differential equation*

$$f(x) = -3 + 2x + 4e^{-x} - e^{-2x}$$

*So that was all very straightforward even if it was involved. Now try your hand at the differential equations in the next frame*

# Revision summary

**1** If  $F(s)$  is the Laplace transform of  $f(x)$  then:

$$L\{f''(x)\} = s^2F(s) - sf(0) - f'(0)$$

$$\text{and } L\{f'''(x)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

**2** Equations of the form:

$$a_nf^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \cdots + a_2f''(x) + a_1f'(x) + a_0f(x) = g(x)$$

where  $a_n, a_{n-1}, \dots, a_2, a_1, a_0$  are constants are called linear, constant-coefficient, inhomogeneous differential equations.

- 3** The Laplace transform can be used to solve constant-coefficient, inhomogeneous differential equations provided  $a_n, a_{n-1}, \dots, a_2, a_1, a_0$  are known constants,  $g(x)$  is a known expression in  $x$ , and the values of  $f(x)$  and its derivatives are known at  $x = 0$ .
- 4** The procedure for solving these equations of second and higher order is the same as that for solving the equations of first order. Namely:
- (a) Take the Laplace transform of both sides of the differential equation
  - (b) Find the expression  $F(s) = L\{f(x)\}$  in the form of an algebraic fraction
  - (c) Separate  $F(s)$  into its partial fractions
  - (d) Find the inverse Laplace transform  $L^{-1}\{F(s)\}$  to find the solution  $f(x)$  to the differential equation.



## Revision exercise

Use the Laplace transform to solve each of the following equations:

(a)  $f'(x) + f(x) = 3$  where  $f(0) = 0$

(b)  $3f'(x) + 2f(x) = x$  where  $f(0) = -2$

(c)  $f''(x) + 5f'(x) + 6f(x) = 2e^{-x}$  where  $f(0) = 0$  and  $f'(0) = 0$

(d)  $f''(x) - 4f(x) = \sin 2x$  where  $f(0) = 1$  and  $f'(0) = -2$

(a)  $f'(x) + f(x) = 3$  where  $f(0) = 0$

Taking Laplace transforms of both sides of the equation gives

$$L\{f'(x)\} + L\{f(x)\} = L\{3\} \text{ so that } [sF(s) - f(0)] + F(s) = \frac{3}{s}$$

$$\text{That is } (s + 1)F(s) = \frac{3}{s} \text{ so } F(s) = \frac{3}{s(s + 1)} = \frac{3}{s} - \frac{3}{s + 1}$$

$$\text{giving the solution as } f(x) = 3 - 3e^{-x} = 3(1 - e^{-x})$$

(b)  $3f'(x) + 2f(x) = x$  where  $f(0) = -2$

Taking Laplace transforms of both sides of the equation gives

$$L\{3f'(x)\} + L\{2f(x)\} = L\{x\} \text{ so that } 3[sF(s) - f(0)] + 2F(s) = \frac{1}{s^2}$$

$$\text{That is } (3s + 2)F(s) - (-6) = \frac{1}{s^2} \text{ so } F(s) = \frac{1 - 6s^2}{s^2(3s + 2)}$$

The partial fraction breakdown gives

$$F(s) = -\frac{3}{4} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s^2} - \frac{15}{4} \cdot \frac{1}{(3s + 2)} = -\frac{3}{4} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s^2} - \frac{5}{4} \cdot \frac{1}{(s + \frac{2}{3})}$$

giving the solution as

$$f(x) = -\frac{3}{4} + \frac{x}{2} - \frac{5e^{-2x/3}}{4}$$

(c)  $f''(x) + 5f'(x) + 6f(x) = 2e^{-x}$  where  $f(0) = 0$  and  $f'(0) = 0$

Taking Laplace transforms of both sides of the equation gives

$$L\{f''(x)\} + L\{5f'(x)\} + L\{6f(x)\} = L\{2e^{-x}\}$$

$$\text{so that } [s^2F(s) - sf(0) - f'(0)] + 5[sF(s) - f(0)] + 6F(s) = \frac{2}{s+1}$$

$$\text{That is } (s^2 + 5s + 6)F(s) = \frac{2}{s+1}$$

$$\text{so } F(s) = \frac{2}{(s+1)(s+2)(s+3)} = \frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3}$$

giving the solution as

$$f(x) = e^{-x} - 2e^{-2x} + e^{-3x}$$

(d)  $f''(x) - 4f(x) = \sin 2x$  where  $f(0) = 1$  and  $f'(0) = -2$

Taking Laplace transforms of both sides of the equation gives

$$L\{f''(x)\} - L\{4f(x)\} = L\{\sin 2x\}$$

$$\text{so that } [s^2F(s) - sf(0) - f'(0)] - 4F(s) = \frac{2}{s^2 + 2^2}$$

$$\text{That is } (s^2 - 4)F(s) - s \cdot 1 - (-2) = \frac{2}{s^2 + 2^2}$$

$$\begin{aligned} \text{so } F(s) &= \frac{2}{(s^2 - 4)(s^2 + 2^2)} + \frac{s - 2}{s^2 - 4} \\ &= \frac{15}{16} \cdot \frac{1}{s+2} + \frac{1}{16} \cdot \frac{1}{s-2} - \frac{1}{8} \cdot \frac{2}{s^2 + 2^2} \end{aligned}$$

giving the solution as

$$f(x) = \frac{15}{16}e^{-2x} + \frac{1}{16}e^{2x} - \frac{\sin 2x}{8}$$

**Example 2.14.** Consider the initial-value problem

$$\frac{d^2 y}{d t^2} + y = 1, \quad y(0) = y'(0) = 0.$$

Let us assume for the moment that the solution  $y = y(t)$  satisfies suitable conditions so that we may invoke (2.22). Taking the Laplace transform of both sides of the differential equation gives

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(1).$$

An application of (2.22) yields

$$s^2 \mathcal{L}(y) - s y(0) - y'(0) + \mathcal{L}(y) = \frac{1}{s},$$

that is,

$$\mathcal{L}(y) = \frac{1}{s(s^2 + 1)}.$$

Writing

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}$$

as a partial fraction decomposition, we find

$$\mathcal{L}(y) = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Applying the inverse transform gives the solution

$$y = 1 - \cos t.$$

One may readily check that this is indeed the solution to the initial-value problem.

Note that the initial conditions of the problem are absorbed into the method, unlike other approaches to problems of this type (i.e., the methods of *variation of parameters* or *undetermined coefficients*).

**General Procedure.** The Laplace transform method for solving ordinary differential equations can be summarized by the following steps.

- (i) Take the Laplace transform of both sides of the equation. This results in what is called the *transformed equation*.
- (ii) Obtain an equation  $\mathcal{L}(y) = F(s)$ , where  $F(s)$  is an algebraic expression in the variable  $s$ .
- (iii) Apply the inverse transform to yield the solution  $y = \mathcal{L}^{-1}(F(s))$ .

The various techniques for determining the inverse transform include partial fraction decomposition, translation, derivative and integral theorems, convolutions, and integration in the complex plane. All of these techniques except the latter are used in conjunction with standard tables of Laplace transforms.

**Example 2.15.** Solve

$$y''' + y'' = e^t + t + 1, \quad y(0) = y'(0) = y''(0) = 0.$$



Taking  $\mathcal{L}$  of both sides gives

$$\mathcal{L}(y''') + \mathcal{L}(y'') = \mathcal{L}(e^t) + \mathcal{L}(t) + \mathcal{L}(1),$$

or

$$\begin{aligned} [s^3 \mathcal{L}(y) - s^2 y(0) - s y'(0) - y''(0)] \\ + [s^2 \mathcal{L}(y) - s y(0) - y'(0)] = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s}. \end{aligned}$$

Putting in the initial conditions gives

$$s^3 \mathcal{L}(y) + s^2 \mathcal{L}(y) = \frac{2s^2 - 1}{s^2(s-1)},$$

which is

$$\mathcal{L}(y) = \frac{2s^2 - 1}{s^4(s+1)(s-1)}.$$

Applying a partial fraction decomposition to

$$\mathcal{L}(y) = \frac{2s^2 - 1}{s^4(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s+1} + \frac{F}{s-1},$$

we find that

$$\mathcal{L}(y) = -\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)},$$

and consequently

$$\begin{aligned} y &= -\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) \\ &= -t + \frac{1}{6}t^3 - \frac{1}{2}e^{-t} + \frac{1}{2}e^t. \end{aligned}$$

In general, the Laplace transform method demonstrated above is particularly applicable to initial-value problems of  $n$ th-order linear ordinary differential equations with constant coefficients, that is,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = f(t), \quad (2.23)$$

$$y(0) = y_0, \quad y'(0) = y_1, \quad \dots, \quad y^{(n-1)}(0) = y_{n-1}.$$

In engineering parlance,  $f(t)$  is known as the *input*, *excitation*, or *forcing function*, and  $y = y(t)$  is the *output* or *response*. In the event the input  $f(t)$  has exponential order and be continuous, the output  $y = y(t)$  to (2.23) can also be shown to have exponential order and be continuous (Theorem A.6). This fact helps to justify the application of the Laplace transform method (see the remark subsequent to Theorem A.6). More generally, when  $f \in L$ , the method can still be applied by assuming that the hypotheses of Theorem 2.12 are satisfied. While the solution  $y = y(t)$  to (2.23) is given by the Laplace transform method for  $t \geq 0$ , it is in general valid on the whole real line,  $-\infty < t < \infty$ , if  $f(t)$  has this domain.

Another important virtue of the Laplace transform method is that the input function  $f(t)$  can be discontinuous.

**Example 2.16.** Solve

$$y'' + y = E u_a(t), \quad y(0) = 0, \quad y'(0) = 1.$$

Here the system is receiving an input of zero for  $0 \leq t < a$  and  $E$  (constant) for  $t \geq a$ . Then

$$s^2 \mathcal{L}(y) - s y(0) - y'(0) + \mathcal{L}(y) = \frac{E e^{-as}}{s}$$

and

$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{s^2 + 1} + \frac{E e^{-as}}{s(s^2 + 1)} \\ &= \frac{1}{s^2 + 1} + E \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-as}.\end{aligned}$$

Whence

$$\begin{aligned}y &= \mathcal{L}^{-1} \left( \frac{1}{s^2 + 1} \right) + E \mathcal{L}^{-1} \left[ \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-as} \right] \\ &= \sin t + E u_a(t) (1 - \cos(t - a)),\end{aligned}$$

by the second translation theorem (1.27). We can also express  $y$  in the form

$$y = \begin{cases} \sin t & 0 \leq t < a \\ \sin t + E(1 - \cos(t - a)) & t \geq a. \end{cases}$$

Note that  $y(a^-) = y(a^+) = \sin a$ ,  $y'(a^-) = y'(a^+) = \cos a$ ,  $y''(a^-) = -\sin a$ , but  $y''(a^+) = -\sin a + E a^2$ . Hence  $y''(t)$  is only piecewise continuous.

**Example 2.17.** Solve

$$y'' + y = \begin{cases} \sin t & 0 \leq t \leq \pi \\ 0 & t > \pi \end{cases} \quad y(0) = y'(\pi) = 0.$$

We have

$$\begin{aligned} s^2 \mathcal{L}(y) + \mathcal{L}(y) &= \int_0^\pi e^{-st} \sin t \, dt \\ &= \frac{-e^{-st}}{s^2 + 1} (s \cdot \sin t + \cos t) \Big|_0^\pi \\ &= \frac{e^{-\pi s}}{s^2 + 1} + \frac{1}{s^2 + 1}. \end{aligned}$$

Therefore,

$$\mathcal{L}(y) = \frac{1}{(s^2 + 1)^2} + \frac{e^{-\pi s}}{(s^2 + 1)^2},$$

and by Example 2.42 (i) and the second translation theorem (1.31),

$$y = \frac{1}{2}(\sin t - t \cos t) + u_{\pi}(t) \left[ \frac{1}{2}(\sin(t - \pi) - (t - \pi) \cos(t - \pi)) \right].$$

In other words,

$$y = \begin{cases} \frac{1}{2}(\sin t - t \cos t) & 0 \leq t < \pi \\ -\frac{1}{2}\pi \cos t & t \geq \pi. \end{cases}$$

Observe that denoting the input function by  $f(t)$ ,

$$\begin{aligned} f(t) &= \sin t(1 - u_{\pi}(t)) \\ &= \sin t + u_{\pi}(t) \sin(t - \pi), \end{aligned}$$

from which

$$\mathcal{L}(f(t)) = \frac{1}{s^2 + 1} + \frac{e^{-\pi s}}{s^2 + 1},$$

again by the second translation theorem.

**General Solutions.** If the initial-value data of (2.23) are unspecified, the Laplace transform can still be applied in order to determine the general solution.

**Example 2.18.** Consider

$$y'' + y = e^{-t},$$

and let  $y(0) = y_0$ ,  $y'(0) = y_1$  be unspecified. Then

$$s^2 \mathcal{L}(y) - s y(0) - y'(0) + \mathcal{L}(y) = \mathcal{L}(e^{-t}),$$

that is,

$$\begin{aligned} \mathcal{L}(y) &= \frac{1}{(s+1)(s^2+1)} + \frac{s y_0}{s^2+1} + \frac{y_1}{s^2+1} \\ &= \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}s - \frac{1}{2}}{s^2+1} + \frac{y_0 s}{s^2+1} + \frac{y_1}{s^2+1}, \end{aligned}$$

by taking a partial fraction decomposition. Applying  $\mathcal{L}^{-1}$ ,

$$y = \frac{1}{2} e^{-t} + \left( y_0 - \frac{1}{2} \right) \cos t + \left( y_1 + \frac{1}{2} \right) \sin t.$$



Since  $y_0, y_1$  can take on all possible values, the general solution to the problem is given by

$$y = c_0 \cos t + c_1 \sin t + \frac{1}{2} e^{-t},$$

where  $c_0, c_1$  are arbitrary real constants. Note that this solution is valid for  $-\infty < t < \infty$ .

**Boundary-Value Problems.** This type of problem is also amenable to solution by the Laplace transform method. As a typical example

**Boundary-Value Problems.** This type of problem is also amenable to solution by the Laplace transform method. As a typical example consider

$$y'' + \lambda^2 y = \cos \lambda t, \quad y(0) = 1, \quad y\left(\frac{\pi}{2\lambda}\right) = 1.$$

Then

$$\mathcal{L}(y'') + \lambda^2 \mathcal{L}(y) = \mathcal{L}(\cos \lambda t),$$

so that

$$(s^2 + \lambda^2) \mathcal{L}(y) = \frac{s}{s^2 + \lambda^2} + s y(0) + y'(0),$$

implying

$$\mathcal{L}(y) = \frac{s}{(s^2 + \lambda^2)^2} + \frac{s y(0)}{s^2 + \lambda^2} + \frac{y'(0)}{s^2 + \lambda^2}.$$

Therefore,

$$y = \frac{1}{2\lambda} t \sin \lambda t + \cos \lambda t + \frac{y'(0)}{\lambda} \sin \lambda t, \quad (2.24)$$

where we have invoked Example 2.42 (ii) to determine the first term and replaced  $y(0)$  with its value of 1. Finally, from (2.24)

$$1 = y\left(\frac{\pi}{2\lambda}\right) = \frac{\pi}{4\lambda^2} + \frac{y'(0)}{\lambda}$$

gives

$$\frac{y'(0)}{\lambda} = 1 - \frac{\pi}{4\lambda^2},$$

and thus

$$y = \frac{1}{2\lambda} t \sin \lambda t + \cos \lambda t + \left(1 - \frac{\pi}{4\lambda^2}\right) \sin \lambda t.$$

Similarly, if the boundary data had been, say

$$y(0) = 1, \quad y' \left( \frac{\pi}{\lambda} \right) = 1,$$

then differentiating in (2.24)

$$y' = \frac{1}{2\lambda} (\sin \lambda t + \lambda t \cos \lambda t) - \lambda \sin \lambda t + y'(0) \cos \lambda t.$$

Thus,

$$1 = y' \left( \frac{\pi}{\lambda} \right) = \frac{-\pi}{2\lambda} - y'(0)$$

and

$$y'(0) = - \left( 1 + \frac{\pi}{2\lambda} \right),$$

to yield

$$y = \frac{1}{2\lambda} t \sin \lambda t + \cos \lambda t - \frac{1}{\lambda} \left( 1 + \frac{\pi}{2\lambda} \right) \sin \lambda t.$$

Sekian dan Terima kasih  
atas perhatian saudara sekalian.