

## SECTION 15.4 Second-Order Nonhomogeneous Linear Equations

## Nonhomogeneous Equations • Method of Undetermined Coefficients • Variation of Parameters

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SOPHIE GERMAIN (1776–1831)

Many of the early contributors to calculus were interested in forming mathematical models for vibrating strings and membranes, oscillating springs, and elasticity. One of these was the French mathematician Sophie Germain, who in 1816 was awarded a prize by the French Academy for a paper entitled "Memoir on the Vibrations of Elastic Plates."

## Nonhomogeneous Equations

In the preceding section, we represented damped oscillations of a spring by the *homogeneous* second-order linear equation

$$\frac{d^2y}{dt^2} + \frac{p}{m} \left( \frac{dy}{dt} \right) + \frac{k}{m} y = 0. \quad \text{Free motion}$$

This type of oscillation is called **free** because it is determined solely by the spring and gravity and is free of the action of other external forces. If such a system is also subject to an external periodic force such as  $a \sin bt$ , caused by vibrations at the opposite end of the spring, the motion is called **forced**, and it is characterized by the *nonhomogeneous* equation

$$\frac{d^2y}{dt^2} + \frac{p}{m} \left( \frac{dy}{dt} \right) + \frac{k}{m} y = a \sin bt. \quad \text{Forced motion}$$

In this section, you will study two methods for finding the general solution of a nonhomogeneous linear differential equation. In both methods, the first step is to find the general solution of the corresponding homogeneous equation.

$$y = y_h \quad \text{General solution of homogeneous equation}$$

Having done this, you try to find a particular solution of the nonhomogeneous equation.

$$y = y_p \quad \text{Particular solution of nonhomogeneous equation}$$

By combining these two results, you can conclude that the general solution of the nonhomogeneous equation is  $y = y_h + y_p$ , as stated in the following theorem.

**THEOREM 15.6** Solution of Nonhomogeneous Linear Equation

Let

$$y'' + ay' + by = F(x)$$

be a second-order nonhomogeneous linear differential equation. If  $y_p$  is a particular solution of this equation and  $y_h$  is the general solution of the corresponding homogeneous equation, then

$$y = y_h + y_p$$

is the general solution of the nonhomogeneous equation.

## Method of Undetermined Coefficients

You already know how to find the solution  $y_h$  of a linear *homogeneous* differential equation. The remainder of this section looks at ways to find the particular solution  $y_p$ . If  $F(x)$  in

$$y'' + ay' + by = F(x)$$

consists of sums or products of  $x^n$ ,  $e^{mx}$ ,  $\cos \beta x$ , or  $\sin \beta x$ , you can find a particular solution  $y_p$  by the method of **undetermined coefficients**. The gist of this method is to guess that the solution  $y_p$  is a generalized form of  $F(x)$ . Here are some examples.

1. If  $F(x) = 3x^2$ , choose  $y_p = Ax^2 + Bx + C$ .
2. If  $F(x) = 4xe^x$ , choose  $y_p = Axe^x + Be^x$ .
3. If  $F(x) = x + \sin 2x$ , choose  $y_p = (Ax + B) + C \sin 2x + D \cos 2x$ .

Then, by substitution, determine the coefficients for the generalized solution.

### EXAMPLE 1 Method of Undetermined Coefficients

Find the general solution of the equation

$$y'' - 2y' - 3y = 2 \sin x.$$

**Solution** To find  $y_h$ , solve the characteristic equation.

$$\begin{aligned} m^2 - 2m - 3 &= 0 \\ (m + 1)(m - 3) &= 0 \\ m &= -1 \quad \text{or} \quad m = 3 \end{aligned}$$

Thus,  $y_h = C_1 e^{-x} + C_2 e^{3x}$ . Next, let  $y_p$  be a generalized form of  $2 \sin x$ .

$$\begin{aligned} y_p &= A \cos x + B \sin x \\ y_p' &= -A \sin x + B \cos x \\ y_p'' &= -A \cos x - B \sin x \end{aligned}$$

Substitution into the original differential equation yields

$$\begin{aligned} y'' - 2y' - 3y &= 2 \sin x \\ -A \cos x - B \sin x + 2A \sin x - 2B \cos x - 3A \cos x - 3B \sin x &= 2 \sin x \\ (-4A - 2B) \cos x + (2A - 4B) \sin x &= 2 \sin x. \end{aligned}$$

By equating coefficients of like terms, you obtain

$$-4A - 2B = 0 \quad \text{and} \quad 2A - 4B = 2$$

with solutions  $A = \frac{1}{5}$  and  $B = -\frac{2}{5}$ . Therefore,

$$y_p = \frac{1}{5} \cos x - \frac{2}{5} \sin x$$

and the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1 e^{-x} + C_2 e^{3x} + \frac{1}{5} \cos x - \frac{2}{5} \sin x. \end{aligned}$$

In Example 1, the form of the homogeneous solution

$$y_h = C_1 e^{-x} + C_2 e^{3x}$$

has no overlap with the function  $F(x)$  in the equation

$$y'' + ay' + by = F(x)$$

However, suppose the given differential equation in Example 1 were of the form

$$y'' - 2y' - 3y = e^{-x}.$$

Now, it would make no sense to guess that the particular solution were  $y = Ae^{-x}$ , because you know that this solution would yield 0. In such cases, you should alter your guess by multiplying by the lowest power of  $x$  that removes the duplication. For this particular problem, you would guess

$$y_p = Axe^{-x}.$$

### EXAMPLE 2 Method of Undetermined Coefficients

Find the general solution of

$$y'' - 2y' = x + 2e^x.$$

**Solution** The characteristic equation  $m^2 - 2m = 0$  has solutions  $m = 0$  and  $m = 2$ . Thus,

$$y_h = C_1 + C_2 e^{2x}.$$

Because  $F(x) = x + 2e^x$ , your first choice for  $y_p$  would be  $(A + Bx) + Ce^x$ . However, because  $y_h$  *already* contains a constant term  $C_1$ , you should multiply the *polynomial part* by  $x$  and use

$$\begin{aligned} y_p &= Ax + Bx^2 + Ce^x \\ y_p' &= A + 2Bx + Ce^x \\ y_p'' &= 2B + Ce^x. \end{aligned}$$

Substitution into the differential equation produces

$$\begin{aligned} y'' - 2y' &= x + 2e^x \\ (2B + Ce^x) - 2(A + 2Bx + Ce^x) &= x + 2e^x \\ (2B - 2A) - 4Bx - Ce^x &= x + 2e^x. \end{aligned}$$

Equating coefficients of like terms yields the system

$$2B - 2A = 0, \quad -4B = 1, \quad -C = 2$$

with solutions  $A = B = -\frac{1}{4}$  and  $C = -2$ . Therefore,

$$y_p = -\frac{1}{4}x - \frac{1}{4}x^2 - 2e^x$$

and the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1 + C_2 e^{2x} - \frac{1}{4}x - \frac{1}{4}x^2 - 2e^x. \end{aligned}$$

In Example 2, the polynomial part of the initial guess

$$(A + Bx) + Ce^x$$

for  $y_p$  overlapped by a constant term with  $y_h = C_1 + C_2e^{2x}$ , and it was necessary to multiply the polynomial part by a power of  $x$  that removed the overlap. The next example further illustrates some choices for  $y_p$  that eliminate overlap with  $y_h$ . Remember that in all cases the first guess for  $y_p$  should match the types of functions occurring in  $F(x)$ .

### EXAMPLE 3 Choosing the Form of the Particular Solution

Determine a suitable choice for  $y_p$  for each of the following.

| $y'' + ay' + by = F(x)$          | $y_h$                                   |
|----------------------------------|---|
| a. $y'' = x^2$                   | $C_1 + C_2x$                            |
| b. $y'' + 2y' + 10y = 4 \sin 3x$ | $C_1e^{-x} \cos 3x + C_2e^{-x} \sin 3x$ |
| c. $y'' - 4y' + 4 = e^{2x}$      | $C_1e^{2x} + C_2xe^{2x}$                |

#### Solution

- a. Because  $F(x) = x^2$ , the normal choice for  $y_p$  would be  $A + Bx + Cx^2$ . However, because  $y_h = C_1 + C_2x$  already contains a linear term, you should multiply by  $x^2$  to obtain

$$y_p = Ax^2 + Bx^3 + Cx^4.$$

- b. Because  $F(x) = 4 \sin 3x$  and each term in  $y_h$  contains a factor of  $e^{-x}$ , you can simply let

$$y_p = A \cos 3x + B \sin 3x.$$

- c. Because  $F(x) = e^{2x}$ , the normal choice for  $y_p$  would be  $Ae^{2x}$ . However, because  $y_h = C_1e^{2x} + C_2xe^{2x}$  already contains an  $xe^{2x}$  term, you should multiply by  $x^2$  to get

$$y_p = Ax^2e^{2x}.$$

### EXAMPLE 4 Solving a Third-Order Equation

Find the general solution of

$$y''' + 3y'' + 3y' + y = x.$$

**Solution** From Example 6 in the preceding section, you know that the homogeneous solution is

$$y_h = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}.$$

Because  $F(x) = x$ , let  $y_p = A + Bx$  and obtain  $y_p' = B$  and  $y_p'' = 0$ . Thus, by substitution, you have

$$(0) + 3(0) + 3(B) + (A + Bx) = (3B + A) + Bx = x.$$

Thus,  $B = 1$  and  $A = -3$ , which implies that  $y_p = -3 + x$ . Therefore, the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} - 3 + x. \end{aligned}$$

## Variation of Parameters

The method of undetermined coefficients works well if  $F(x)$  is made up of polynomials or functions whose successive derivatives have a cyclic pattern. For functions such as  $1/x$  and  $\tan x$ , which do not have such characteristics, it is better to use a more general method called **variation of parameters**. In this method, you assume that  $y_p$  has the same *form* as  $y_h$ , except that the constants in  $y_h$  are replaced by variables.

### Variation of Parameters

To find the general solution to the equation  $y'' + ay' + by = F(x)$ , use the following steps.

1. Find  $y_h = C_1y_1 + C_2y_2$ .
2. Replace the constants by variables to form  $y_p = u_1y_1 + u_2y_2$ .
3. Solve the following system for  $u_1'$  and  $u_2'$ .

$$\begin{aligned}u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= F(x)\end{aligned}$$

4. Integrate to find  $u_1$  and  $u_2$ . The general solution is  $y = y_h + y_p$ .

### EXAMPLE 5 Variation of Parameters

Solve the differential equation

$$y'' - 2y' + y = \frac{e^x}{2x}, \quad x > 0.$$

**Solution** The characteristic equation  $m^2 - 2m + 1 = (m - 1)^2 = 0$  has one solution,  $m = 1$ . Thus, the homogeneous solution is

$$y_h = C_1y_1 + C_2y_2 = C_1e^x + C_2xe^x.$$

Replacing  $C_1$  and  $C_2$  by  $u_1$  and  $u_2$  produces

$$y_p = u_1y_1 + u_2y_2 = u_1e^x + u_2xe^x.$$

The resulting system of equations is

$$\begin{aligned}u_1'e^x + u_2'xe^x &= 0 \\ u_1'e^x + u_2'(xe^x + e^x) &= \frac{e^x}{2x}.\end{aligned}$$

Subtracting the second equation from the first produces  $u_2' = 1/(2x)$ . Then, by substitution in the first equation, you have  $u_1' = -\frac{1}{2}$ . Finally, integration yields

$$u_1 = -\int \frac{1}{2} dx = -\frac{x}{2} \quad \text{and} \quad u_2 = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} \ln x = \ln \sqrt{x}.$$

From this result it follows that a particular solution is

$$y_p = -\frac{1}{2}xe^x + (\ln \sqrt{x})xe^x$$

and the general solution is

$$y = C_1e^x + C_2xe^x - \frac{1}{2}xe^x + xe^x \ln \sqrt{x}.$$

**EXAMPLE 6** Variation of Parameters

Solve the differential equation

$$y'' + y = \tan x.$$

**Solution** Because the characteristic equation  $m^2 + 1 = 0$  has solutions  $m = \pm i$ , the homogeneous solution is

$$y_h = C_1 \cos x + C_2 \sin x.$$

Replacing  $C_1$  and  $C_2$  by  $u_1$  and  $u_2$  produces

$$y_p = u_1 \cos x + u_2 \sin x.$$

The resulting system of equations is

$$\begin{aligned} u_1' \cos x + u_2' \sin x &= 0 \\ -u_1' \sin x + u_2' \cos x &= \tan x. \end{aligned}$$

Multiplying the first equation by  $\sin x$  and the second by  $\cos x$  produces

$$\begin{aligned} u_1' \sin x \cos x + u_2' \sin^2 x &= 0 \\ -u_1' \sin x \cos x + u_2' \cos^2 x &= \sin x. \end{aligned}$$

Adding these two equations produces  $u_2' = \sin x$ , which implies that

$$\begin{aligned} u_1' &= -\frac{\sin^2 x}{\cos x} \\ &= \frac{\cos^2 x - 1}{\cos x} \\ &= \cos x - \sec x. \end{aligned}$$

Integration yields

$$\begin{aligned} u_1 &= \int (\cos x - \sec x) dx \\ &= \sin x - \ln |\sec x + \tan x| \end{aligned}$$

and

$$\begin{aligned} u_2 &= \int \sin x dx \\ &= -\cos x \end{aligned}$$

so that

$$\begin{aligned} y_p &= \sin x \cos x - \cos x \ln |\sec x + \tan x| - \sin x \cos x \\ &= -\cos x \ln |\sec x + \tan x| \end{aligned}$$

and the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1 \cos x + C_2 \sin x - \cos x \ln |\sec x + \tan x|. \end{aligned}$$

## EXERCISES FOR SECTION 15.4

In Exercises 1–4, verify the solution of the differential equation.

| <i>Solution</i>                                  | <i>Differential Equation</i> |
|--|------------------------------|
| 1. $y = 2(e^{2x} - \cos x)$                      | $y'' + y = 10e^{2x}$         |
| 2. $y = (2 + \frac{1}{2}x)\sin x$                | $y'' + y = \cos x$           |
| 3. $y = 3 \sin x - \cos x \ln  \sec x + \tan x $ | $y'' + y = \tan x$           |
| 4. $y = (5 - \ln  \sin x )\cos x - x \sin x$     | $y'' + y = \csc x \cot x$    |

In Exercises 5–20, solve the differential equation by the method of undetermined coefficients.

5.  $y'' - 3y' + 2y = 2x$
6.  $y'' - 2y' - 3y = x^2 - 1$
7.  $y'' + y = x^3$
8.  $y'' + 4y = 4$
9.  $y'' + 2y' = 2e^x$
10.  $y'' - 9y = 5e^{3x}$
11.  $y'' - 10y' + 25y = 5 + 6e^x$
12.  $16y'' - 8y' + y = 4(x + e^x)$
13.  $y'' + y' = 2\sin x$
14.  $y'' + y' - 2y = 3 \cos 2x$
15.  $y'' + 9y = \sin 3x$
16.  $y'' + 4y' + 5y = \sin x + \cos x$
17.  $y''' - 3y' + 2y = 2e^{-2x}$
18.  $y''' - y'' = 4x^2$
19.  $y' - 4y = xe^x - xe^{4x}$
20.  $y' + 2y = \sin x$

## 21. Think About It

- (a) Explain how, by observation, you know that a particular solution of the differential equation  $y'' + 3y = 12$  is  $y_p = 4$ .
- (b) Use the explanation of part (a) to give a particular solution of the differential equation  $y'' + 5y = 10$ .
- (c) Use the explanation of part (a) to give a particular solution of the differential equation  $y'' + 2y' + 2y = 8$ .

## 22. Think About It

- (a) Explain how, by observation, you know that a form of a particular solution of the differential equation  $y'' + 3y = 12 \sin x$  is  $y_p = A \sin x$ .
- (b) Use the explanation of part (a) to find a particular solution of the differential equation  $y'' + 5y = 10 \cos x$ .
- (c) Compare the algebra required to find particular solutions in parts (a) and (b) with that required if the form of the particular solution were  $y_p = A \cos x + B \sin x$ .

In Exercises 23–28, solve the differential equation by the method of variation of parameters.

23.  $y'' + y = \sec x$
24.  $y'' + y = \sec x \tan x$
25.  $y'' + 4y = \csc 2x$
26.  $y'' - 4y' + 4y = x^2 e^{2x}$
27.  $y'' - 2y' + y = e^x \ln x$
28.  $y'' - 4y' + 4y = \frac{e^{2x}}{x}$

**Electrical Circuits** In Exercises 29 and 30, use the electrical circuit differential equation

$$\frac{d^2q}{dt^2} + \left(\frac{R}{L}\right)\frac{dq}{dt} + \left(\frac{1}{LC}\right)q = \left(\frac{1}{L}\right)E(t)$$

where  $R$  is the resistance (in ohms),  $C$  is the capacitance (in farads),  $L$  is the inductance (in henrys),  $E(t)$  is the electromotive force (in volts), and  $q$  is the charge on the capacitor (in coulombs). Find the charge  $q$  as a function of time for the electrical circuit described. Assume that  $q(0) = 0$  and  $q'(0) = 0$ .

29.  $R = 20$ ,  $C = 0.02$ ,  $L = 2$   
 $E(t) = 12 \sin 5t$
30.  $R = 20$ ,  $C = 0.02$ ,  $L = 1$   
 $E(t) = 10 \sin 5t$

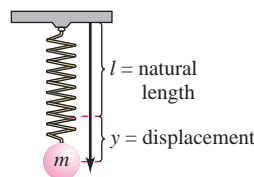


**Vibrating Spring** In Exercises 31–34, find the particular solution of the differential equation

$$\frac{w}{g}y''(t) + by'(t) + ky(t) = \frac{w}{g}F(t)$$

for the oscillating motion of an object on the end of a spring. Use a graphing utility to graph the solution. In the equation,  $y$  is the displacement from equilibrium (positive direction is downward) measured in feet, and  $t$  is time in seconds (see figure). The constant  $w$  is the weight of the object,  $g$  is the acceleration due to gravity,  $b$  is the magnitude of the resistance to the motion,  $k$  is the spring constant from Hooke's Law, and  $F(t)$  is the acceleration imposed on the system.

31.  $\frac{24}{32}y'' + 48y = \frac{24}{32}(48 \sin 4t)$   
 $y(0) = \frac{1}{4}$ ,  $y'(0) = 0$
32.  $\frac{2}{32}y'' + 4y = \frac{2}{32}(4 \sin 8t)$   
 $y(0) = \frac{1}{4}$ ,  $y'(0) = 0$
33.  $\frac{2}{32}y'' + y' + 4y = \frac{2}{32}(4 \sin 8t)$   
 $y(0) = \frac{1}{4}$ ,  $y'(0) = -3$
34.  $\frac{4}{32}y'' + \frac{1}{2}y' + \frac{25}{2}y = 0$   
 $y(0) = \frac{1}{2}$ ,  $y'(0) = -4$



Spring displacement

- 35. Vibrating Spring** Rewrite  $y_h$  in the solution for Exercise 31 by using the identity

$$a \cos \omega t + b \sin \omega t = \sqrt{a^2 + b^2} \sin(\omega t + \phi)$$

where  $\phi = \arctan a/b$ .

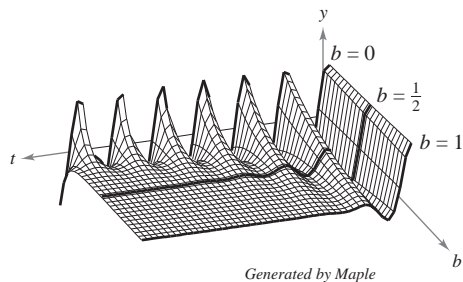
- 36. Vibrating Spring** The figure shows the particular solution of the differential equation

$$\frac{4}{32}y'' + by' + \frac{25}{2}y = 0$$

$$y(0) = \frac{1}{2}, y'(0) = -4$$

for values of the resistance component  $b$  in the interval  $[0, 1]$ . (Note that when  $b = \frac{1}{2}$ , the problem is identical to that of Exercise 34.)

- If there is no resistance to the motion ( $b = 0$ ), describe the motion.
- If  $b > 0$ , what is the ultimate effect of the retarding force?
- Is there a real number  $M$  such that there will be no oscillations of the spring if  $b > M$ ? Explain your answer.



- 37. Parachute Jump** The fall of a parachutist is described by the second-order linear differential equation

$$\frac{w}{g} \frac{d^2y}{dt^2} - k \frac{dy}{dt} = w$$

where  $w$  is the weight of the parachutist,  $y$  is the height at time  $t$ ,  $g$  is the acceleration due to gravity, and  $k$  is the drag factor of the parachute. If the parachute is opened at 2000 feet,  $y(0) = 2000$ , and at that time the velocity is  $y'(0) = -100$  feet per second, then for a 160-pound parachutist, using  $k = 8$ , the differential equation is

$$-5y'' - 8y' = 160.$$

Using the given initial conditions, verify that the solution of the differential equation is

$$y = 1950 + 50e^{-1.6t} - 20t.$$

- 38. Parachute Jump** Repeat Exercise 37 for a parachutist who weighs 192 pounds and has a parachute with a drag factor of  $k = 9$ .

- 39.** Solve the differential equation

$$x^2y'' - xy' + y = 4x \ln x$$

given that  $y_1 = x$  and  $y_2 = x \ln x$  are solutions of the corresponding homogeneous equation.

- 40. True or False?**  $y_p = -e^{2x} \cos e^{-x}$  is a particular solution of the differential equation

$$y'' - 3y' + 2y = \cos e^{-x}.$$