the reflecting horizon, and is known as a horizontal event. On the other hand the arrival time of the diffracted pulse will be given by

$$t^2 - x/v^2 = vT_0^2,$$

where x is the horizontal distance from the shot-receiver location to the diffracting edge, and v is the wave speed. Therefore, on the seismic section, the diffracted pulse will appear as a hyperbolic trajectory, whose apex determines the edge of the truncated horizon. The amplitude of the reflection event is much greater than that of the diffraction event. Therefore, it is often extremely difficult to observe the diffraction in seismic reflection data, unless an amplitude correction function is applied.

[1]T. Young, Philos. Trans. R. Soc. London **12** (1802); T. Young, in *Miscellaneous Works of the Late Thomas Young, Vol. 1* (Murray, London, England, 1855), p. 140.

[2]J. B. Keller, J. Opt. Soc. Am. **52**, 116 (1962).

[3]J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi, *Electromagnetic and Acoustical Scattering by Simple Shapes* (North-Holland, Amsterdam, 1969), pp. 308–345.

[4]H. S. Carslaw, Proc. London Math. Soc. **30**, 121 (1899).

[5]P. C. Clemmow, Q. J. Mech. Appl. Math. **3**, 241 (1950).

On the radiation from point charges

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An alternative derivation of the radiation field of a point charge is presented. It starts with the Fourier components of the current produced by the moving charge. The electric field is found from the vector wave equation. Each step in the integration permits physical interpretation. The retarded time appears very naturally in this derivation. The interpretation of the present derivation is that a charge at constant velocity $\bar{v}(|\bar{v}| < c)$ does not radiate, not because it is unaccelerated, but because it has no Fourier components synchronous with waves traveling at the speed of light. Of course, Cherenkov radiation in a medium, in which the velocity of electromagnetic propagation is less than c, is the classic example of radiation by a charge moving at constant velocity.

I. INTRODUCTION

The equations of macroscopic electrodynamics attribute the rate of energy generation to the current density $\bar{J}(\bar{r}, t)$. This fact is underscored by Poynting's theorem, in which the power delivered per unit volume is equated to the scalar product of the electric field and current density, $\bar{E}(\bar{r}, t) \cdot \bar{J}(\bar{r}, t)$. An oscillating dipole in the steady state radiates power that is equal to the time average of $\int \bar{E} \cdot \bar{J}\, dv$ integrated over the current distribution. On the other hand, it is well known that a single charged particle in uniform motion does not radiate in free space. The far field of a single particle derived from the Liénard–Wiechert potentials, which decays with distance from the particle as $1/r$ and thus accounts for radiation, is proportional to the acceleration of the particle. Thus the Poynting vector of the radiation fields is proportional to $[\dot{\bar{J}}]^2$. This is one dilemma that faces the student of electrodynamics when he first sees the derivation from the Liénard–Wiechert potentials. Another new concept facing the student is the "retarded time" in the derivation of the Liénard–Wiechart potentials which does not enter explicitly into the discussion of macroscopic electrodynamics.

The usual derivation of the Liénard–Wiechert potentials is not conceptually simple.[1,2] It rests on the superposition of the fields produced by different volume elements of the charge distribution representing the radiating particle, which is treated for this purpose, and this purpose alone, as occupying a volume of finite size. An alternative derivation presented by Jackson[3] rests on a careful evaluation of integrals of the spatial delta function. This derivation has much to recommend it. However, it does not provide a physical picture for the meaning of the mathematical development. Sommerfeld[4] has presented two derivations of the potentials produced by a moving particle, both based on the Green's function and contour integration in the complex plane. His derivation is particularly elegant. Yet none of these three derivations bring out the dichotomy between $\bar{J}(\bar{r}, t)$ as responsible for power transfer to the field accord ing to Poynting's theorem, and $\dot{\bar{J}}(\bar{r}, t)$ as responsible for radiation.

This paper is the outgrowth of work on the noise radiation from a "wiggled" electron beam in a waveguide[5] (free electron laser). The radiation in a waveguide is analyzed most naturally in terms of the excitation of waveguide modes. The electric field of such a mode is proportional to the appropriate modal component of $\bar{J}$ (not $\omega\bar{J}$). Yet when the analysis is taken to the limit of an infinite waveguide cross section, the same result is obtained as with the Liénard–Wiechert potentials.[6] This suggests alternate interpretations of radiation of a moving particle.

In Sec. II we summarize the spatial Fourier representation of a moving particle. In Sec. III we solve the vector wave equation in Fourier transform space ($\bar{k}$, ω space).

Then we perform the inverse Fourier transform in two steps. First, the integral is carried over the magnitude of $\bar{k}$. At this point, the expression assumes a particularly appealing form—the field is proportional to $\bar{J}_\perp(\frac{\omega}{c}\bar{n},\omega)$, where $\bar{J}_\perp(\bar{k},\omega)$ is the Fourier transform of the current perpendicular to $\bar{k}$ and $\bar{n}\equiv\bar{k}/|\bar{k}|$. Further, this field appears summed over all the modes per unit volume and solid angle by the weighting factor

$$\rho(\omega,\Omega)d\omega d\Omega=\frac{4\pi\omega^2}{c^2}d\left(\frac{\omega}{c}\right)\frac{d\Omega}{4\pi}.$$

The integral over all angles brings in a factor of ω, i.e., $\omega\bar{J}_\perp$ can now be interpreted as containing the acceleration (but not evaluated with respect to retarded time). The analysis brings in the retarded time naturally, no factor $1-(\bar{v}\cdot\bar{n})/c$ appears in the expression for $\bar{E}_\perp(\bar{r},t)$. Finally, we obtain the well-known far field as a function of $\bar{r}$ and t.[3]

II. THE SOURCE AND ITS FOURIER TRANSFORMS

Consider a charged particle of charge q and position $\bar{r}_0(t)$. The charge density of the particle is described by

$$\rho(\bar{r},t)=q\delta[\bar{r}-\bar{r}_0(t)] \quad (1)$$

where $\delta(\bar{r}-\bar{r}_0)$ is the spatial unit impulse function. The current density is

$$\bar{J}(\bar{r},t)=q\dot{\bar{r}}_0(t)\delta[\bar{r}-\bar{r}_0(t)]\,. \quad (2)$$

The spatial Fourier transform represents the current density as a superposition of spatial exponentials, $\exp-i\bar{k}\cdot\bar{r}$.

$$\bar{J}(\bar{k},t)=\int\int\int d^3\bar{r}q\dot{\bar{r}}_0(t)\delta[\bar{r}-\bar{r}_0(t)]e^{-i\bar{k}\cdot\bar{r}}$$
$$=q\dot{\bar{r}}_0(t)e^{-i\bar{k}\cdot\bar{r}_0}. \quad (3)$$

The full space-time Fourier transform is of course,

$$\bar{J}(\bar{k},\omega)=\int\int\int\int dt\,d^3\bar{r}\,\bar{J}(\bar{r},t)e^{-i\bar{k}\cdot\bar{r}}e^{i\omega t}. \quad (4)$$

The inverse Fourier transform is

$$\bar{J}(\bar{r},t)=\left(\frac{1}{2\pi}\right)^4\int d\omega\int\int\int d\bar{k}^3\bar{J}(\bar{k},\omega)e^{-i\omega t}e^{i\bar{k}\cdot\bar{r}}. \quad (5)$$

III. THE ELECTROMAGNETIC FIELD

The electric field obeys the vector wave equation

$$\nabla\times(\nabla\times\bar{E})+\frac{1}{c^2}\frac{\partial^2\bar{E}}{\partial t^2}=-\mu_0\frac{\partial\bar{J}}{\partial t}. \quad (6)$$

The space-time Fourier transform of the vector wave equation is

$$\bar{k}\times[\bar{k}\times\bar{E}(\bar{k},\omega)]+\frac{\omega^2}{c^2}\bar{E}(k,\omega)$$
$$=-i\omega\mu_0\bar{J}(\bar{k},\omega)\,. \quad (7)$$

In the far field, only the component perpendicular to $\bar{k}$ is of interest. Concentrating on this component one has

$$\bar{E}_\perp(\bar{k},\omega)=\frac{-i\omega\mu_0\bar{n}\times[\bar{n}\times\bar{J}(\bar{k},\omega)]}{k^2-(\omega^2/c^2)}, \quad (8)$$

with

$$\bar{n}=\frac{\bar{k}}{|\bar{k}|}. \quad (9)$$

IV. THE INVERSE SPATIAL FOURIER TRANSFORM

The inverse space-time Fourier transform involves the integrals

$$\int\frac{d\omega}{2\pi}e^{-i\omega t}\left(\frac{1}{2\pi}\right)^3\int\int\int d^3\bar{k}e^{i\bar{k}\cdot\bar{r}}.$$

We shall retain the Fourier transform with respect to time and thus not carry out the integration over ω. But we shall focus on a spectral width $d\omega$ of the field and thus write down expressions for $\bar{E}_\perp(\bar{r},\omega)(d\omega/2\pi)$. We separate the integrals into an integral over the magnitude of $\bar{k}$, and into a double integral with respect to the angles θ and ϕ of $\bar{k}$ with respect to $\bar{r}$:

$$\bar{E}_\perp(\bar{r},\omega)\frac{d\omega}{2\pi}=-\frac{d\omega}{2\pi}\left(\frac{1}{2\pi}\right)^3\int\int d\phi d\theta\sin\theta$$
$$\times\int i\omega\mu_0k^2dk\frac{\bar{n}\times[\bar{n}\times\bar{J}(\bar{k},\omega)]}{k^2-(\omega^2/c^2)}e^{i\bar{k}\cdot\bar{r}}. \quad (10)$$

The last integral can be carried out by contour integration. For $\bar{k}\cdot\bar{r}>0$, the contour must be closed into the negative imaginary half plane of k with the result

$$\bar{E}_\perp(\bar{r},\omega)\frac{d\omega}{2\pi}=\left(\frac{1}{2\pi}\right)^2\frac{\omega^2}{c^2}d\left(\frac{\omega}{c}\right)\int\int\frac{d\phi d\theta\sin\theta}{4\pi}$$
$$\times\sqrt{\frac{\mu_0}{\epsilon_0}}c\bar{n}\times\left[\bar{n}\times\bar{J}\left(\frac{\omega}{c}\bar{n},\omega\right)\right]e^{i(\omega/c)\bar{n}\cdot\bar{r}}. \quad (11)$$

This expression may be rewritten in a way that lends itself to an appealing interpretation. The density of (linearly polarized) modes per unit volume and unit solid angle $\rho(\omega,\Omega)$ is

$$\rho(\omega,\Omega)d\omega d\Omega=\frac{1}{2\pi}\left(\frac{\omega}{c}\right)^2d\left(\frac{\omega}{c}\right)\frac{d\Omega}{4\pi}. \quad (12)$$

With this definition, one has

$$\bar{E}_\perp(\bar{r},\omega)\frac{d\omega}{2\pi}=\frac{c}{2\pi}\int\rho(\omega,\Omega)d\omega d\Omega\sqrt{\frac{\mu_0}{\epsilon_0}}$$
$$\times\bar{n}\times\left[\bar{n}\times\bar{J}\left(\frac{\omega}{c}\bar{n},\omega\right)\right]e^{i(\omega/c)\bar{n}\cdot\bar{r}}. \quad (13)$$

The field $\bar{E}_\perp(\bar{r},\omega)(d\omega/2\pi)$ is proportional to $-\bar{J}_\perp((\omega/c)\bar{n},\omega)$ namely, the Fourier component for which $k=\omega/c$. Factors of ω that multiply the Fourier component of the current are due to the density of modes per unit volume and unit solid angle. An unaccelerated charge does not radiate in free space, not because it experiences no acceleration, but because it has no Fourier component

$$\bar{J}\left(\frac{\omega}{c}\bar{n},\omega\right).$$

Indeed, from (3)

$$J(\bar{k},\omega)=\int dtq\bar{v}e^{-i\bar{k}\cdot\bar{v}t+i\omega t}$$
$$=2\pi q\bar{v}\delta(\omega-\bar{k}\cdot\bar{v})\,. \quad (14)$$

The only nonzero Fourier components are for

$$k=\frac{\omega}{v\cos\theta}>\frac{\omega}{c}, \quad (15)$$

where θ is the angle between $\bar{v}$ and $\bar{k}$. The reason for the radiation of an accelerated charge is that the Fourier decomposition of the current acquires Fourier components that are "synchronous" with the light velocity, i.e., with the propagation constant $|\bar{k}| = \omega/c$. Thus, for example, an oscillating charge

$$\bar{r}_0(t) = \bar{d}\sin\omega_0 t\,, \tag{16}$$

has a Fourier spectrum

$$\bar{J}(\bar{k},\omega) = \frac{q\omega_0 d}{2} J_m(k\cos\theta d)\{\delta[\omega-(m+1)\omega_0] + \delta[\omega-(m-1)\omega_0]\}\,, \tag{17}$$

where the J_m s are Bessel functions of order m. These Fourier components can, and do, acquire phase velocities that are equal to the light velocity. For small kd only $m=0$ remains and is approximately independent of k, $J_0[(\omega_0/c)\cos\theta d] \simeq 1$.

V. INTEGRATION OVER ANGLES

Starting with (11), we note that the exponential is a strong function of θ, whereas the component $\bar{n}\times[\bar{n}\times\bar{J}]$ varies much more slowly and thus can be pulled out from under the integration. We have to integrate an expression of the form

$$\frac{1}{2\pi}\frac{\omega^2 d\omega}{c^3}\int_0^{2\pi}\int_0^{\pi}\frac{d\phi d\theta\sin\theta}{4\pi}e^{i(\omega/c)\cos\theta r} = -\frac{1}{2}i\frac{\omega}{c^2 r}\frac{d\omega}{2\pi}e^{i(\omega/c)r}\,,$$

where the upper limit on θ is ignored because of the rapid variation of the exponent. With this result introduced in (11) one has

$$E_\perp(\bar{r},\omega)\frac{d\omega}{2\pi} = -\frac{d\omega}{2\pi}\frac{i}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\omega}{cr}\bar{n} \times\left[\bar{n}\times\bar{J}\left(\frac{\omega}{c}\bar{n},\omega\right)\right]e^{i(\omega/c)r}. \tag{18}$$

Here, $\bar{n}$ is the direction of the radius vector $\bar{r}$. We note now that a factor of ω appears in front of the current. One may therefore interpret the source as containing the acceleration although one must emphasize that multiplication by $-i\omega$ represents differentiation with respect to the time coordinate, not with respect to retarded time. The two are related by the factor $1-\bar{n}\cdot\bar{v}/c$. It seems more natural to attribute the factor to the integration over all the modes, in particular because then Cherenkov radiation presents less of a mystery. Cherenkov radiation is produced by an unaccelerated particle, but since the velocity of light is less than c, the particle current can have Fourier components synchronous with $\omega/c\sqrt{\epsilon/\epsilon_0}$, where ϵ is the dielectric constant of the medium.

If one introduces (3), one finds from (18)

$$E_\perp(\bar{r},\omega)\frac{d\omega}{2\pi} = -\frac{iq}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\omega}{cr}\left\{\int dt'\bar{n}\times[\bar{n}\times\dot{\bar{r}}_0(t')] \times\exp+i\omega\left(t'+\frac{r}{c}-\bar{n}\cdot\frac{\bar{r}_0(t')}{c}\right)\right\}\frac{d\omega}{2\pi}. \tag{19}$$

This expression has in it the retarded time

$$t_{\rm ret} = t+\frac{r}{c}-\frac{\bar{n}\cdot\bar{r}_0(t)}{c}. \tag{20}$$

It does not contain the factor $1-(\bar{n}\cdot\bar{v}/c)$ which appears in the Liénard–Wiechert fields. However, we deal here with the Fourier transform in time of $\bar{E}$. In the next section we show that the usual expression for the far field is obtained from the Fourier transform of (19).[3]

VI. THE ELECTRIC FIELD IN SPACE-TIME

The electric field in space-time is obtained by the inverse Fourier transform of (19)

$$E_\perp(\bar{r},t) = \int_{-\infty}^{\infty}E_\perp(\bar{r},\omega)e^{-i\omega t}\frac{d\omega}{2\pi} = -\frac{q}{4\pi}\sqrt{\mu_0/\epsilon_0}\frac{1}{r} \times\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int dt'\bar{n}[\bar{n}\times\dot{\bar{r}}_0(t')]i\omega \times\exp-i\omega\left(t-t'-\frac{r}{c}-\bar{n}\cdot\frac{\bar{r}_0(t')}{c}\right). \tag{21}$$

The factor $i\omega$ can be replaced by a differentiation of the exponential

$$\frac{d}{dt'}\exp-i\omega\left(t-t'-\frac{r}{c}-\bar{n}\cdot\frac{\bar{r}_0(t')}{c}\right) = i\omega\left(1-\frac{\bar{n}\cdot\dot{\bar{r}}_0}{c}\right)\exp-i\omega\left(t-t'-\frac{r}{c}-\bar{n}\cdot\frac{\bar{r}_0(t')}{c}\right). \tag{22}$$

In the notation of Jackson, we designate by κ the factor

$$1-\frac{\bar{n}\cdot\dot{\bar{r}}_0}{c} = \kappa\,, \tag{23}$$

and, by $\bar{\beta}$ the velocity normalized by the speed of light, $\dot{\bar{r}}_0/c$. Introducing (22) and (23) into (21), and integrating by parts over t', one has

$$\bar{E}_\perp(\bar{r},t) = \frac{q}{4\pi}\sqrt{\mu_0/\epsilon_0}\frac{1}{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int dt' \times\exp-i\omega\left(t-t'-\frac{r}{c}-\frac{\bar{n}\cdot\bar{r}_0/t'}{c}\right) \times\frac{d}{dt'}\left(\frac{\bar{n}\times[\bar{n}\times\bar{\beta}(t')]}{\kappa}\right). \tag{24}$$

In the far field, the time derivative of $\bar{n}$ is negligible compared with the derivative of $\bar{v}$ and $\bar{r}_0$. In this limit, it is easy to show that[3]

$$\frac{d}{dt'}\left(\frac{\bar{n}\times[\bar{n}\times\bar{\beta}(t')]}{\kappa}\right) = \frac{\bar{n}\times[(\bar{n}-\bar{\beta})\times\dot{\bar{\beta}}]}{\kappa^2}. \tag{25}$$

The integral over all ω of

$$\frac{1}{2\pi}\exp-i\omega\left(t-t'-\frac{r}{c}-\frac{\bar{n}\cdot\bar{r}_0(t')}{c}\right),$$

produces the delta function

$$\delta\left(t-t'-\frac{r}{c}-\frac{\bar{n}\cdot\bar{r}_0(t')}{c}\right).$$

Introducing these results into (24), gives

$$E_\perp(\bar{r},t)=\frac{q}{4\pi}\sqrt{\mu_0/\epsilon_0}\,\frac{1}{r}\int_{-\infty}^{\infty}dt'$$
$$\times\delta\left(t-t'-\frac{r}{c}-\frac{\bar{n}\cdot\bar{r}_0(t')}{c}\right)$$
$$\times\frac{\bar{n}\times[(\bar{n}-\bar{\beta})\times\dot{\bar{\beta}}]}{\kappa^2}. \qquad (26)$$

When the argument of the delta function is a function of the variable of integration, the integration gives[3]

$$\int g(x)\delta[f(x)-\alpha]dx=\left(\frac{g(x)}{df/dx}\right)_{f(x)=\alpha}. \qquad (27)$$

In the present case

$$f(t')=t'+\frac{r}{c}+\frac{\bar{n}\cdot\bar{r}_0(t')}{c} \qquad (28)$$

and

$$\frac{df}{dt'}=1-\bar{n}\cdot\bar{\beta}=\kappa. \qquad (29)$$

Thus we have from (26) the final result

$$\bar{E}_\perp(\bar{r},t)=\frac{q}{4\pi}\sqrt{\mu_0/\epsilon_0}\,\frac{1}{r\kappa^3}\{\bar{n}\times[(\bar{n}-\bar{\beta})\times\dot{\bar{\beta}}]\}_{\text{ret}}. \qquad (30)$$

The expression in brackets is evaluated at

$$t-\frac{r}{c}-\frac{\bar{n}\cdot\bar{r}_0(t)}{c},$$

the retarded time. Equation (30) is the standard result, here written in mks units.

VII. CONCLUSIONS

When the radiation field of a charged particle is evaluated by its inverse Fourier transform, the expressions resulting from successive integrals permit physically appealing interpretations. The integral over the magnitude of the propagation constant gives the Fourier component of the field that has the appearance (13)

$$\bar{E}_\perp(\bar{r},\omega)\frac{d\omega}{2\pi}=\frac{c}{2\pi}\int\int\rho(\omega,\Omega)d\omega d\Omega\sqrt{\frac{\mu_0}{\epsilon_0}}\,\bar{n}$$
$$\times\left[\bar{n}\times\bar{J}\left(\frac{\omega}{c}\bar{n},\omega\right)\right]e^{i(\omega/c)\bar{n}\cdot\bar{r}}. \qquad (13)$$

Thus, radiation is produced by the Fourier component of $\bar{J}_\perp$ that has $\bar{k}=(\omega/c)\bar{n}$, i.e., that travels with the speed of light. Particles that are unaccelerated do not radiate, because their current density has no Fourier components synchronous with the light velocity. Thus the presence of Cherenkov radiation of particles moving in "slow" media, i.e., with light velocities in the medium less than c, is a natural consequence of this interpretation of (13) and does not present any conceptual difficulties.

The integral over the mode density produces a factor of ω multiplying $\bar{J}_\perp$. Thus, one could interpret the expression for $\bar{E}_\perp(\bar{r},\omega)(d\omega/2\pi)$ as caused by acceleration, except of course that a multiplication by $-i\omega$ results via differentiation with respect to time—not retarded time. The Fourier $\bar{E}_\perp(\bar{r},\omega)$ contains an integrand expressed in terms of retarded time which came in quite naturally through the preceding integrations.

Equation (13) also makes plausible the appearance of $\bar{E}\cdot\bar{J}$ in Poynting's theorem. The $\bar{E}$ field of the plane wave associated with a plane wave of current $\bar{J}_\perp(\bar{k},\omega)$, is proportional to $\bar{J}_\perp$, not $\omega\bar{J}_\perp$ and 180 deg out of phase as required by power conservation. Factors of ω are the result of the superposition of plane waves at different angles of propagation. Inverse Fourier transformation of $E_\perp(\bar{r},\omega)$ gives the usual expression for the far field in space-time. The derivation by this approach has much to recommend it because the successive mathematical steps permit physical interpretation. Further, the quantum analysis of photon detection[7] uses as its starting part expressions that can be shown to be contained in the present analysis.

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[1]W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1956).
[2]R. D. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1979), 6th printing, pp. 21–5.
[3]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[4]Ph. Frank, R. v. Mises, *Die Differential- and Integralgleichungen der Mechanik und Physik, Zweiter physikalischer Teil* (Vieweg und Sohn, Braunschweig, 1935).
[5]H. A. Haus and M. N. Islam, J. Appl. Phys. **54**, 4784 (1983).
[6]W. B. Colson, IEEE J. Quantum Electron. **QE-17**, 1417 (1981).
[7]R. J. Cook, Phys. Rev. A **25**, 2164 (1982).