An Economic-Statistical Design of $\bar{x}$ Control Charts with Multiple Assignable Causes

Fong-Jung Yu$^*$  Ching-Shih Tsou$^2$  Kai-I Huang$^3$  Zhang Wu$^4$

$^1$Department of Industrial Engineering and Technology Management, Da-Yeh University, Taiwan
$^2$Institute of Information Science and Management, National Taipei College of Business, Taiwan
$^3$College of Management, Tunghai University, Taiwan
$^4$School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore

(Received 05/2009; Revised 11/2009; Accepted 03/2010)

Abstract

Duncan first proposed the economic design of $\bar{x}$ control charts in 1956 to control normal process means and ensure that an economic design control chart actually lowers the cost, compared with a Shewhart control chart. Many authors have studied the control charts from economic viewpoint from then on. An economic design does not consider the statistical properties, such as type I or type II error and average time to signal (ATS). To improve these issues, an economic-statistical design of control charts has been developed under the consideration of one assignable cause. However, there are multiple assignable causes in the real practice such as machine problem, material deviation, human errors, etc. In order to have a real application, this research will extend the original research from single to multiple assignable causes to establish an economic-statistical model of $\bar{x}$ control chart. A numerical example is used to illustrate the performance of the proposed model and to compare the lost cost between the pure economic and economic-statistical control chart design. A sensitivity analysis is also conducted in this numerical example.

Keywords: SPC, $\bar{x}$ control chart, multiple assignable causes, economic design, economic-statistical design

$^*$ Correspondence: Department of Industrial Engineering and Technology Management, Da-Yeh University
No. 168, University Rd., Dacon, Changhua County 51591, Taiwan, R.O.C.
E-mail: fischer@mail.dyu.edu.tw
1. INTRODUCTION

A control chart is one of the most important techniques of statistical process control. The sample means of measurements of a quality characteristic in samples taken from the production process are plot on this chart. It can be applied to reduce the variability of a process and also used to maintain a process in a state of statistical control. It is a very useful process monitoring technique. A sample size $n$, a sampling interval between samples $h$, and a coefficient of control limits $k$ (as a multiple of standard deviation) must be selected by engineers or analysts for a use of control charts. Traditionally, control charts are designed with respect to statistical criteria only. Its performance may be unsatisfactory from the economic viewpoint. Duncan (1956) first proposed an economic model of $\bar{x}$ control charts to control normal process means and insure that the economic design control chart actually has a lower cost, compared with a Shewhart control chart. In Duncan’s model (1956), it assumes that there is only one assignable cause in the production process and the failure mechanism from in-control to out-of-control state is exponential distribution. Four cost items are considered in this model including a sampling and testing cost, an increasing cost from out-of-control process, a false alarm cost and a cost for searching and removing the problems. This research was the stimulus for much of the subsequent studies on the subject.

For example, multiple assignable causes are more practical in the production process than single one. Duncan (1971) extended his research from single to multiple assignable causes in which the failure mechanism was also assumed to follow an exponential distribution. Montgomery and Heikes (1976) considered the probability distribution with an increasing failure rate. It is more realistic and extended to non-Markovian process. Saniga (1977) proposed an economic design of the joint $\bar{x}$ and $R$ control charts for two assignable causes. One assignable cause results in a shift of the process mean and the other results in a shift of the process variance. The time for occurrence of assignable cause is also exponentially distributed. In order to simplify the complexity of the analysis model, Lorenzen and Vance (1986) proposed a general approach to determine the economic design parameters of control charts. It is applicable regardless of the statistic used and only requires the calculation of the average run-length of the statistics. On the other hand, a revised approach is explored by extending the Duncan’s model (1956) from an exponential to a Weibull distribution (Banerjee and Rahim, 1988). Ho and Case (1994) provided a literature review of such models covering the period 1981-1991. Wu (1996) presented an approach to determine the optimum control limits of the $\bar{x}$ chart for skewed process distributions. Wu et al. (2002) proposed a new algorithm for designing the $\bar{x}$ and S Charts for Monitoring Process Capability. Yu and Wu (2004) proposed an Economic Design for Variable Sampling Interval (VSI) MA Control Charts. Yu and Chen (2005) provided an Economic Design for a VSI $\bar{x}$ Control Chart for a Continuous-Flow Process. Yu and Hou (2006) provided an Economic Design for a VSI $\bar{x}$ Control Chart with multiple assignable causes. Wu et al. (2007) proposed an algorithm for deploying manpower to a Statistical Process Control (SPC) system that monitors a multistage manufacturing system. This algorithm minimizes the expected total cost by optimizing the amount of allocated manpower in the SPC system. Christopher et al. (2010) proposed the economic design of $\bar{x}$ control charts with continuously variable sampling intervals to the field in that it allows the sampling interval to be determined by the extremity of the most recent sample. Engin (2009) extended Duncan’s economic control chart design
methodology as an alternative way for estimating and optimizing machine efficiency in the case of multi-machine assignments.

The weakness of economic design is due to the fact that it does not consider statistical properties such as the probability of type I and type II errors when selecting the parameters for the control chart. Statistically designed control charts have desirable statistical properties, but the operating cost can be high. Saniga (1989) proposed a method to economically design control charts that have bounds of the probabilities on type I and type II errors and the average time to signal (ATS). This design can be viewed as the improvement of economic or statistical design, since they consider economic factors while achieving desirable statistical properties. This study will extend Duncan’s (1971) model from economic to economic-statistical model to provide more protection to both producers and consumers. A numerical example is also employed to illustrate the model’s performance and to demonstrate its utility.

2. THE METHOD OF ECONOMIC STATISTICAL DESIGN OF CONTROL CHARTS

Statistical design of control charts aims at the optimal chart performance in terms of statistical properties. An economic design of control charts is based on an economic criterion. An economic statistical design of a control chart can be defined as that the economic-loss cost function is minimized subject to the constrained maximum values of probabilities of type I and type II errors, as well as the maximum value of Average Time to Signal (ATS) when a process shifts. On the basis of the specified statistical constraints, control charts are then designed to have long ATS\(_0\) values when the process is in control and small ATS\(_1\) values when the process is out of control (Saniga, 1989; Yang, 1998). Let \(Y\) be the set of design parameters and \(L\) be the expected hourly loss cost function of an \(\bar{X}\) control chart economic model. Then, the economic statistical model of the \(\bar{X}\) control chart can be formulated as:

\[
\begin{align*}
\text{minimize} & \quad L(Y) \\
\text{subject to} & \quad \alpha \leq \alpha_u, \\
& \quad \beta \leq \beta_u
\end{align*}
\]

where \(\alpha_u\) and \(\beta_u\) are the desired bounds on the type I error probability and type II error probability, respectively. The solution of this model is an improvement to the economic design because both the statistical properties and minimization of loss cost have been considered. A solution without the constraints is the optimum economic design of the control charts (Yang, 1998; Yu et al., 2006).

3. DEFINITION AND ASSUMPTIONS

The features to be studied in this article are as follows (Duncan, 1971):

1. The process is either in-control or out-of-control state only and is in-control state at the beginning.
2. The \( m \)th assignable cause will produce a shift in the process mean of \( \delta_m \sigma \) where \( \sigma \) is the standard deviation of \( X \).
3. The standard deviation is assumed to remain invariant when process shifts.
4. The probability of positive or negative shift will be the same.
5. The failure rate of the \( m \)th assignable cause follows an exponential distribution with \( \lambda_m \).
6. The distribution of \( \bar{X} \) is normal.
7. Production is continuous during the search and repair.
8. The detection probability when assignable cause occurs is greater than \( 1 - \beta_m \).
9. The type I error of the control chart is less than \( \alpha \).
10. The out-of-control ATS will be less than 4.

At this time, there are an expected cycle time and an expected loss to be formulated in this model construction.

4. EXPECTED CYCLE TIME IN PRODUCTION PROCESS

The cycle length consists of four time intervals, namely, (1) the interval during which the process is in control; (2) the interval during which the process is out-of-control but still undetected; (3) the time required to sample, inspect, evaluate and plot a sample mean; (4) the time required to search and repair for the assignable cause. On the basis of Duncan’s (1971) model, an economic design of an \( \bar{X} \) control chart is considered when there are multiple assignable causes. It is assumed that the process starts in a state of statistical control with mean \( \mu_0 \) and standard deviation \( \sigma \). The \( \bar{X} \) chart is to detect multiple assignable causes from the beginning. The average time for occurrence of the \( m \)th assignable cause is equal to \( 1/\lambda_m \) \((m=1, 2, \ldots)\). There is no limit on the value of \( m \) for the process and it is also free from the occurrence of other assignable cause. Then the probability that no assignable cause has occurred at the end of time \( t \) is \( e^{-t \sum \lambda_m} \). Therefore, the mean time during which the process is in-control can be shown as:

\[
T_0 = \frac{1}{\Lambda}
\]  

\((2)\)

where \( \Lambda = \sum \lambda_m \). The process mean will shift to \( \mu_0 + \delta_m \sigma \) or \( \mu_0 - \delta_m \sigma \) when the \( m \)th assignable cause occurs. Let \( P_m \) be the probability that a sample point falls outside the control limits after the occurrence of the \( m \)th cause. Then \( P_m \) can be shown as:

\[
P_m = 1 - \Phi \left( k - \delta_m \sqrt{n} \right) + \Phi \left( -k - \delta_m \sqrt{n} \right)
\]  

\((3)\)

where \( \Phi(\bullet) \) is the cumulative density function (CDF) of the standard normal distribution, \( n \) is the sample size and \( k \) is the coefficient of the control limit. Then the average sampling number that will be
taken after the \( m \)th assignable cause has occurred will be \( 1 / P_m \). And the time interval between the assignable cause occurrence and detection can be shown as the following equation (Duncan, 1971):

\[
T_m = h \left(1 / P_m \right)
\]

where \( h \) is the sampling interval.

Suppose samples are taken at intervals of \( h \) hours, and the \( m \)th assignable cause occurs in the interval between the \( j \)th and \((j+1)\)th samples. Then the mean time of occurrence of the \( m \)th assignable cause within an interval between samples is:

\[
\tau_m = \int_{j}^{(j+1)h} \frac{\lambda_m (t - jh) e^{-\lambda t} dt}{\int_{j}^{(j+1)h} \lambda_m e^{-\lambda t} dt}
\]

Suppose the time for testing and analyzing for each sample is \( g \) and the average time required to find and repair the \( m \)th assignable cause is \( D_m \) after it really occurs. Then a complete average cycle-length, therefore, will be:

\[
T = T_0 + \frac{1}{\Lambda} \left( \sum_{m=1}^{\infty} \lambda_m \left( T_m - \tau_m + gn + D_m \right) \right)
\]

Let \( \alpha \) be the probability of type I error. Then

\[
\alpha = 2\Phi(-k)
\]

The expected number of false alarms per complete average cycle-length before any assignable cause occurs will be \( \alpha \) times the expected sampling number taken in the in-control period, and can be expressed as:

\[
N = \alpha \sum_{j=0}^{\infty} \int_{j}^{(j+1)h} j \lambda e^{-\lambda t} dt
\]

**5. THE EXPECTED LOSS COST GENERATED**

The expected loss cost of control chart in this model during a cycle consists of the penalty cost due to poor quality incurred when the assignable cause occurs, the loss-cost of actually locating and repairing the assignable cause after obtaining an out-of-control signal, the loss-cost per unit time to identify the false alarms per cycle and the expected cost for sampling and maintaining control chart. It can be expressed as the following equation:
An Economic-Statistical Design of \( \bar{X} \) Control Charts with Multiple Assignable Causes

\[
L = \left[ \sum_{m=1}^{M} \frac{\lambda_m}{\Lambda} \left( (T_m - \tau_m + gn + D_m) + M_m + W_m \right) \right] / (b + cn) / h
\]  \hspace{1cm} (9)

where \( M_m \) is a penalty cost due to poor quality when the \( m \)th assignable cause occurs, \( V \) is the average cost per false alarm, and \( W_m \) is the locating and repairing cost when assignable cause occurs. Finally, \( b \) and \( c \) are the fixed and variable costs per sample for testing and plotting. The fixed cost is independent of sample size.

The goal of the economic-statistical design of \( \bar{X} \) control charts is to find the optimal values of the design parameters, \( n, h \) and \( k \), in order to minimize the loss-cost function \( L \) in equation (9) under the statistical constraint in equation (1). The loss-cost function \( L \) is a very complicated function of the decision variables, \( n, h \) and \( k \). In this research, the Grid search (McWilliams, 1994) technique is used and modified to search the optimal solution.

6. A NUMERICAL EXAMPLE

In this section, Duncan’s (1971) data were employed to illustrate the performance of the proposed model. Here it was supposed that there were twelve assignable causes in the process. The parameter \( \lambda_m \) is the average rate of occurrence per unit time of the \( m \)th assignable cause which, when it occurs, produces a shift in the process mean of \( \delta_m \sigma \). Hence, given the occurrence of the assignable cause, the ratio \( \lambda_m / \Lambda \) (with \( \Lambda = \sum \lambda_m \)) is the conditional occurrence probability of the \( m \)th assignable cause. When the process shift occurs, let the magnitudes of the shift \( (\delta_m \sigma) \) be \( 0.75 \sigma, 1.25 \sigma, 1.75 \sigma, 2.25 \sigma, 2.75 \sigma, 3.25 \sigma, 3.75 \sigma, 4.25 \sigma, 4.75 \sigma, 5.25 \sigma, 5.75 \sigma \) and \( 6.25 \sigma \), respectively. In addition, the occurrence rate of the \( m \)th assignable cause \( (\lambda_m) \) is assigned proportional to \( 0.5 \times \exp(-0.5 \times \delta_m^2) \), and this approach is referred to as the negative exponential prior distribution for \( \delta_m \sigma \) (Duncan, 1971). The magnitudes of the shifts \( \delta_m \) also affect other cost factors including \( D_m, M_m \) and \( W_m \), which will vary along with the magnitude of the shift \( \delta_m \).

Detailed data are shown in Table 1.

The time \( g \) for sample testing and analyzing for each sample is 0.05. When an assignable cause occurred, substantial manpower might have been required, and the production rate was often decreased until the cause was found and repaired. When a false alarm signal appeared, its cost was estimated to be \( V = 25 \). The cost of maintaining and plotting the control chart was \( b = 1 \) per \( n \) sample unit; and was \( c = 0.1 \) per sample unit for the analysis.

Suppose the first bounds for the statistical constraints in the probability of Type I error \( (\alpha) \), the probability of Type II error \( (\beta) \) are \( \alpha = 0.01, \beta = 0.10 \). Another bounds \( \alpha = 0.05, \beta = 0.08 \) are arbitrarily chosen for comparison. Table 2 shows that the relative design parameters and the costs for pure economic design, economic statistical design. Table 2 reveals that the sample sizes is 3 for pure economic and more than four times the sample size to 14 or 12 for economic statistical design in two different constraint sets, respectively. It also shows that to achieve the desired statistical properties we need to pay a price for it. The increase in overall expected cost is about 22.12%
Table 1. Input values of the parameters

<table>
<thead>
<tr>
<th>Item</th>
<th>$\delta_m$</th>
<th>$\lambda_m$</th>
<th>$M_m$</th>
<th>$W_m$</th>
<th>$D_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.75</td>
<td>0.001098</td>
<td>7.22</td>
<td>19.68</td>
<td>4.17</td>
</tr>
<tr>
<td>2</td>
<td>1.25</td>
<td>0.000855</td>
<td>27.60</td>
<td>14.57</td>
<td>3.08</td>
</tr>
<tr>
<td>3</td>
<td>1.75</td>
<td>0.000666</td>
<td>76.14</td>
<td>11.81</td>
<td>2.50</td>
</tr>
<tr>
<td>4</td>
<td>2.25</td>
<td>0.000519</td>
<td>165.69</td>
<td>9.84</td>
<td>2.08</td>
</tr>
<tr>
<td>5</td>
<td>2.75</td>
<td>0.000404</td>
<td>302.36</td>
<td>9.06</td>
<td>1.92</td>
</tr>
<tr>
<td>6</td>
<td>3.25</td>
<td>0.000314</td>
<td>433.64</td>
<td>8.66</td>
<td>1.84</td>
</tr>
<tr>
<td>7</td>
<td>3.75</td>
<td>0.000245</td>
<td>570.32</td>
<td>8.37</td>
<td>1.77</td>
</tr>
<tr>
<td>8</td>
<td>4.25</td>
<td>0.000191</td>
<td>659.86</td>
<td>8.17</td>
<td>1.72</td>
</tr>
<tr>
<td>9</td>
<td>4.75</td>
<td>0.000148</td>
<td>708.40</td>
<td>8.05</td>
<td>1.70</td>
</tr>
<tr>
<td>10</td>
<td>5.25</td>
<td>0.000115</td>
<td>728.97</td>
<td>7.93</td>
<td>1.68</td>
</tr>
<tr>
<td>11</td>
<td>5.75</td>
<td>0.000090</td>
<td>735.78</td>
<td>7.83</td>
<td>1.66</td>
</tr>
<tr>
<td>12</td>
<td>6.25</td>
<td>0.000070</td>
<td>737.56</td>
<td>7.73</td>
<td>1.64</td>
</tr>
</tbody>
</table>

Table 2. Loss cost of $\tau$ control chart

<table>
<thead>
<tr>
<th>Designs</th>
<th>$n$</th>
<th>$h$</th>
<th>$k$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>ATS</th>
<th>Loss cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>ED</td>
<td>3</td>
<td>1.58</td>
<td>2.65</td>
<td>0.0080</td>
<td>0.4002</td>
<td>2.6290</td>
<td>3.9751</td>
</tr>
<tr>
<td>ESD*</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>0.0096</td>
<td>0.0999</td>
<td>2.5996</td>
<td>4.8545</td>
</tr>
<tr>
<td>ESD**</td>
<td>12</td>
<td>2.48</td>
<td>2.16</td>
<td>0.0308</td>
<td>0.0797</td>
<td>2.6947</td>
<td>4.8874</td>
</tr>
</tbody>
</table>

Note: ED : pure economic design; ESD: economic statistical design;
* : ESD is under $\alpha \leq 0.01, \beta \leq 0.1, \text{ATS} \leq 4$;
** : ESD is under $\alpha \leq 0.05, \beta \leq 0.08, \text{ATS} \leq 4$;

$((4.8545 - 3.9751)/ 3.9751)$ and 22.97% $((4.8874 - 3.9751)/ 3.9751)$ from pure economic design to economic statistical design. In the economic-statistical design, the probabilities of type I error ($\alpha$) are all within the desired limits, and the test power ($1-\beta$) is improved from 59.98% to 90.01% (1-0.0999) and 92.03% (1-0.0797).

7. SENSITIVITY ANALYSIS

This section discusses the robustness of the model when the parameters of times, costs, shifts and failure rates vary. The values of the parameters given in the example are used as the basic case, so that a unique cost item or input parameters such as $\delta_m$, $\lambda_m$, $M_m$, $W_m$, $D_m$, $b$ and $c$ are changed by $\pm 10\%$, $\pm 25\%$ and $\pm 50\%$ from the original values to determine the trend in the minimum loss-cost under the same statistical constrains. For each of the $6 \times 7$ alternative cases, the optimal values of $n$, $h$ and $k$ are determined, and are shown in Table 3. This analysis also gives an
Table 3. Sensitivity analysis

<table>
<thead>
<tr>
<th></th>
<th>%</th>
<th>n</th>
<th>h</th>
<th>k</th>
<th>Loss cost</th>
<th>% of original cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$ change</td>
<td>100</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8545</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>56</td>
<td>3.82</td>
<td>2.59</td>
<td>8.2062</td>
<td>169.04</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>25</td>
<td>2.79</td>
<td>2.59</td>
<td>5.8083</td>
<td>119.65</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>18</td>
<td>2.5</td>
<td>2.65</td>
<td>5.1953</td>
<td>107.02</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>12</td>
<td>2.23</td>
<td>2.64</td>
<td>4.6570</td>
<td>95.93</td>
</tr>
<tr>
<td></td>
<td>125</td>
<td>9</td>
<td>2.1</td>
<td>2.59</td>
<td>4.3891</td>
<td>90.41</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>8</td>
<td>1.96</td>
<td>2.98</td>
<td>4.2111</td>
<td>86.75</td>
</tr>
<tr>
<td>$\lambda$ change</td>
<td>100</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8545</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>14</td>
<td>3.27</td>
<td>2.59</td>
<td>2.9216</td>
<td>60.18</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>14</td>
<td>2.69</td>
<td>2.59</td>
<td>3.9204</td>
<td>80.76</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>14</td>
<td>2.46</td>
<td>2.59</td>
<td>4.4870</td>
<td>92.43</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>14</td>
<td>2.23</td>
<td>2.59</td>
<td>5.2152</td>
<td>107.43</td>
</tr>
<tr>
<td></td>
<td>125</td>
<td>14</td>
<td>2.1</td>
<td>2.59</td>
<td>5.7451</td>
<td>118.35</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>14</td>
<td>1.93</td>
<td>2.59</td>
<td>6.6030</td>
<td>136.02</td>
</tr>
<tr>
<td>$M$ change</td>
<td>100</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8545</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>14</td>
<td>3.32</td>
<td>2.59</td>
<td>4.8250</td>
<td>99.39</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8398</td>
<td>99.70</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8486</td>
<td>99.88</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8604</td>
<td>100.12</td>
</tr>
<tr>
<td></td>
<td>125</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8693</td>
<td>100.30</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8840</td>
<td>100.61</td>
</tr>
<tr>
<td>$W$ change</td>
<td>100</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8545</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8250</td>
<td>99.39</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8398</td>
<td>99.70</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8486</td>
<td>99.88</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8604</td>
<td>100.12</td>
</tr>
<tr>
<td></td>
<td>125</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8693</td>
<td>100.30</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8840</td>
<td>100.61</td>
</tr>
<tr>
<td>$D$ change</td>
<td>100</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8545</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>14</td>
<td>2.32</td>
<td>2.59</td>
<td>3.9571</td>
<td>81.51</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>14</td>
<td>2.33</td>
<td>2.59</td>
<td>4.4072</td>
<td>90.79</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>14</td>
<td>2.33</td>
<td>2.59</td>
<td>4.6760</td>
<td>96.32</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>5.0326</td>
<td>103.67</td>
</tr>
<tr>
<td></td>
<td>125</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>5.2990</td>
<td>109.16</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>14</td>
<td>2.35</td>
<td>2.59</td>
<td>5.7407</td>
<td>118.26</td>
</tr>
<tr>
<td>$b/c$ change</td>
<td>100</td>
<td>14</td>
<td>2.34</td>
<td>2.59</td>
<td>4.8545</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>14</td>
<td>1.72</td>
<td>2.59</td>
<td>4.2632</td>
<td>87.82</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>14</td>
<td>2.05</td>
<td>2.59</td>
<td>4.5812</td>
<td>94.37</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>14</td>
<td>2.23</td>
<td>2.59</td>
<td>4.7494</td>
<td>97.83</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>14</td>
<td>2.44</td>
<td>2.59</td>
<td>4.9549</td>
<td>102.07</td>
</tr>
<tr>
<td></td>
<td>125</td>
<td>14</td>
<td>2.59</td>
<td>2.59</td>
<td>5.0980</td>
<td>105.02</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>14</td>
<td>2.82</td>
<td>2.59</td>
<td>5.3196</td>
<td>109.58</td>
</tr>
</tbody>
</table>
indication of the sensitivity of the loss cost to each of the input parameters. When the magnitudes of all the shifts \( \delta_m \) vary from 0.5, 0.75, 0.9, 1.1, 1.25, to 1.5 times the original value, the estimated minimum cost will be $8.2062, $5.8083, $5.1965, \ldots, $4.2111 at different sampling interval \((h)\) and control coefficient \((k)\). The ratio between these costs and the original minimum cost ($4.8545) is 169.04% ($8.2062/$4.8545) to 86.75% ($4.2111/$4.8545). The less the \( \delta_m \) value is, the larger the sampling sizes are. The sampling intervals will decrease when the \( \delta_m \) value increases. Other interesting observations pertaining to the economic-statistical charts are the following:

1. When the rate of occurrence of assignable causes \((\lambda)\) increases, the values of \(h\) decrease and the loss-cost increases. That is, the higher the rate at which assignable causes occur, the shorter the sampling interval. The loss-cost changes from 60.18% to 136.02% compared with the original cost. But the values on \(k\) still remain unchanged due to upper bounds of type I and II errors probabilities.

2. The loss-cost will be 60.20% to 136.61% of the original one if penalty cost \((M)\) due to poor quality changes and it has almost the same effect as \(\lambda\). The smaller the \(M\), the larger the \(h\).

3. The search and repair cost \((W)\) have only a negligible effect on \(h\) or \(k\), whereby the loss-cost changes from 99.39% to 100.61% of the original.

4. Similarly, the searching and repairing time \((D)\) have only a negligible effect on \(h\) or \(k\), whereby the loss-cost changes from 81.51% to 118.26%.

5. The time required to sample, test, evaluate and plot a sample \((b/c)\) also has a little effect on \(h\). The higher the cost of \(b\) and \(c\), the longer of sampling intervals, whereby the loss-cost changes from 87.82% to 109.58%.

6. It is necessary to pay more attention to the parameters of the magnitude of shift \((\delta)\), the rate of occurrence of assignable causes \((\lambda)\) and the penalty cost \((M)\) when the process is out-of-control.

8. CONCLUSIONS

In practice, multiple assignable causes are more realistic than the single ones. For the economic design of the control charts, single assignable cause is more popular than multiple ones. Also the economic design of control chart does not consider statistical properties, such as the probabilities of type I and type II errors when selecting design parameters. In order to provide more protection for both consumers and producers, an economic-statistical model of an \(\bar{X}\) control chart with multiple assignable causes has been constructed to determine the optimal design parameters in this paper. To compare and verify the effectiveness of the proposed and pure economic models, a numerical example is also provided and it reveals that statistical performance of control charts can be improved significantly and achieved the desired statistical requirements by using an economic-statistical design with an acceptable increase in the cost. A sensitivity analysis has shown that the parameters of magnitude of shift \((\delta)\), the penalty cost \((M)\) when the process is out-of-control, and the rate of occurrence of assignable causes \((\lambda)\) should receive more attention for estimating the parameters for loss cost calculation.
Nomenclature

1. Design variables
   \( n \)  Sample size
   \( h \)  sampling intervals
   \( k \)  Control limit expressed in units of standard deviation

2. Parameters related to assignable cause
   \( \mu_0 \)  Target mean
   \( \sigma \)  True process standard deviation
   \( \delta_m \)  Magnitude of an assignable cause expressed in units of \( \sigma \)
   \( \lambda_m \)  Occurrence rate of an assignable cause per unit time

3. Cost and technical parameters
   \( D_m \)  Average time taken to find and repair an assignable cause after detection
   \( g \)  average time for sampling, testing and analyzing
   \( M_{\mu} \)  Income reduction when \( \mu = \mu_0 + \delta_m \sigma \)
   \( W_m \)  Average cost of looking for and repairing an assignable cause when one does exist
   \( V \)  Average cost of looking for an assignable cause when a false alarm occurs
   \( b \)  Fixed cost per sampling, inspecting, evaluating and plotting
   \( c \)  Variable cost per sampling, inspecting, evaluating and plotting

Acknowledgements

The authors wish to express their appreciation for the support by National Sciences Council of the Republic of China, Grant No. NSC-95-2221-E-212-047.

References

Banerjee, P. K. and Rahim, M. A., 1988, Economic design of \( \bar{x} \) control charts under Weibull shock models, Technometrics, 30(4), 407-414.

Christopher, C. A., Kros, J. F. and Said, S. E., 2010, Economic design of \( \bar{x} \) control charts with continuously variable sampling intervals, Quality and Reliability Engineering International, 26(3), 235-245.

Duncan, A. J., 1956, The economic design of \( \bar{x} \) charts used to maintain current control of a process, Journal of the American Statistical Association, 51(274), 228-242.

Duncan, A.J., 1971, The economic design of \( \bar{x} \) charts when there is a multiplicity of assignable causes, Journal of the American Statistical Association, 66(333), 107-121.

Engin, A. B. 2009, Using economic \( \bar{x} \) control chart design methodology to estimate and optimize machine efficiency in the case of multimachine assignments, Applied Stochastic Models in Business and Industry, Published online in Wiley InterScience.


An Economic-Statistical Design of $\bar{x}$ Control Charts with Multiple Assignable Causes

多重失效平均值管制圖經濟統計設計

余豐榮 $^1$ 鄭慶士 $^2$ 黃開義 $^3$ Zhang Wu $^4$

$^1$大葉大學工業工程與科技管理系
$^2$台北商業技術學院資訊科學與管理研究所
$^3$東海大學管理學院
$^4$新加坡南洋理工大學機械與宇航工程學院

摘 要

Duncan 於 1956 年首先提出平均值管制圖經濟設計並證實經濟設計管制圖比傳統修華特管制圖有較低之品質成本。之後，有多位學者以成本之角度來探討管制圖之設計。經濟設計管制圖並未考量管制圖之統計特性，如型一或型二誤差及錯誤訊號平均出現之時間等。為改善此一現象，單一失效之管制圖統計經濟設計已經被提出。然而，在實際製程中，通常有機械問題、材料變異、人為錯誤等多重失效存在。為較符合實際應用需求，本研究延伸單一失效為多重失效，以建立平均值管制圖統計經濟設計模式。同時以數值案例說明模式之應用，並比較經濟設計與統計經濟設計之成本差異，且執行敏感度分析以了解設計參數對管制圖成本之影響。

關鍵詞: 統計製程管制、平均值管制圖、多重失效、經濟設計、統計經濟設計

* 聯絡作者: 大葉大學工業工程與科技管理系，51591 彰化縣大村鄉學府路 168 號。
  E-mail: fischer@mail.dyu.edu.tw