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Subgrid Scale Stress Models for the Large-Eddy Simulation of Rotating Turbulent Flows

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The modeling of the subgrid scale stresses is considered from a theoretical standpoint with a view toward developing models that are more suitable for the large-eddy simulation of rotating turbulent flows. It is proven, as a rigorous consequence of the Navier–Stokes equations, that such models must be generally invariant under the extended Galilean group and must be frame-indifferent in the limit of two-dimensional turbulence which can be approached in a rapidly rotating framework. Furthermore, it is shown that a significant increase in the rotation rate must be accompanied by a substantial reduction in the energy dissipation rate of the turbulence. Vorticity subgrid scale stress models as well as several other commonly used models are shown to be in serious violation of one or more of these constraints and, hence, are not generally suitable for the description of rotating flows. Alternative models with the correct physical properties are discussed and compared.

1. INTRODUCTION

During the past decade, some significant progress has been made in the understanding of the physics of turbulent flows by the use of large-eddy simulations. In this approach, the small scale turbulence is modeled through the subgrid scale stresses while the large scale structures are calculated directly (cf. Deardorff, 1970; Ferziger, 1977; Biringen and Reynolds, 1981; Moin and Kim, 1982). Since the large eddies of turbulence depend strongly on the flow configuration in question and are responsible for most of the production and redistribution of energy, it can be argued that a basic understanding
of the physics of turbulence will best be attained through their direct calculation. The small scales of turbulence are more universal serving mainly as a source for dissipation and, hence, are more amenable to modeling.

The purpose of the present paper is to examine in more detail the physical constraints that the Navier–Stokes equations place on the structure of subgrid scale stress models in rotating turbulent flows in order to obtain improved models for such large-eddy simulations. It will be proven, as a rigorous consequence of the Navier–Stokes equations, that subgrid scale stress models must be invariant under the extended Galilean group and must be frame-indifferent in the limit of two-dimensional turbulence which can be approached in a rapidly rotating framework sufficiently far from solid boundaries. In fact, provided that the disparity in the time scales of the subgrid scale and mean motions is large (as would be expected on physical grounds), frame-indifference will be shown to be valid in a strong approximate sense for more general turbulent flows. It will also be proven that a significant increase in the rotation rate of the fluid must be accompanied by a substantial reduction in the energy dissipation rate of the turbulence and, hence, an enhancement of non-dissipative momentum exchanges between the large and small scales.

Vorticity subgrid scale stress models (cf. Ferziger, 1977) as well as a variety of other commonly used models (cf. Biringen and Reynolds, 1981; Kim, 1983) will be shown to be in serious violation of these physical constraints. However, by making a small modification in the recent linear combination model proposed by Bardina, Ferziger, and Reynolds (1983) (the Bardina constant must be adjusted from a value of 1.1 to 1), it will be shown that these physical constraints can be satisfied identically. Likewise, it will also be shown that the subgrid scale stress model proposed by Clark, Ferziger, and Reynolds (1979) is consistent with these physical constraints although it does have certain disadvantages in comparison to the linear combination model. Further modeling improvements which are needed will also be discussed along with the prospects for future research.

2. SUBGRID SCALE STRESS MODELS

We will consider the turbulent flow of a homogeneous and incom-
pressible Newtonian fluid governed by the Navier–Stokes equations

\[
\rho [\partial u/\partial t + (u \cdot \nabla)u] = -\nabla p + \mu \nabla^2 u,
\]

and continuity equation

\[
\nabla \cdot u = 0.
\]

In (1) and (2), \( u \) is the velocity vector, \( p \) is the pressure function, \( \rho \) is the density of the fluid, and \( \mu \) is the dynamic viscosity of the fluid. Here, we have assumed that the body forces are conservative. In the large-eddy simulations, any flow variable \( \phi \) is decomposed into a mean and fluctuating part as follows

\[
\phi = \bar{\phi} + \phi'.
\]

Here, \( \bar{\phi} \) is given by

\[
\bar{\phi} = \int_{D} G(x' - x, \Delta) \phi(x') d^3 x',
\]

where \( G(x' - x, \Delta) \) is a filter function that constitutes a Dirac delta sequence (see Deardorff, 1970), \( \Delta \) is the computational mesh size, and \( D \) is the fluid domain. As a direct consequence of the Riemann–Lebesgue theorem, (4) substantially reduces the amplitude of the high frequency Fourier components in space of any flow variable \( \phi \). Consequently, \( \bar{\phi} \) can be more accurately termed the large scale part and \( \phi' \) the small scale (or subgrid scale) part of \( \phi \). Unlike in the more traditional Reynolds averaging,

\[
\bar{\phi} \neq \bar{\bar{\phi}},
\]

and hence,

\[
\bar{\phi}' \neq 0
\]

in general.

The equations of motion for the large eddies of turbulence, which are obtained by filtering (1) and (2), are as follows

\[
\rho [\partial \bar{u}/\partial t + (\bar{u} \cdot \nabla)\bar{u}] = -\nabla \bar{p} + \mu \nabla^2 \bar{u} - \rho \nabla \cdot \tau,
\]
where $\tau$ is the subgrid scale stress tensor whose components are given by

$$\tau_{kl} = \overline{\bar{u}_k \bar{u}_l} - \overline{u_k u_l} + \overline{u'_k u'_l} + \overline{u'_k u'_l}. \quad (9)$$

Here, $\tau$ can be decomposed as follows:

$$\tau = L + C + R, \quad (10)$$

where

$$L_{kl} = \overline{\bar{u}_k \bar{u}_l} - \overline{u_k u_l}, \quad (11)$$

$$C_{kl} = \overline{u'_k u'_l}, \quad R_{kl} = \overline{u'_k u'_l}. \quad (12, 13)$$

are, respectively, the Leonard stresses, the subgrid scale cross stresses, and the subgrid scale Reynolds stresses. Since only the Leonard stresses can be calculated directly, the equations of motion for the large eddies are not closed. Closure is achieved by supplementing Eqs. (7) and (8) with subgrid scale stress models of the general form

$$\tau(x, t) = [\tau(x', t'); x, t], \quad x' \in D, \ t' \in (-\infty, t). \quad (14)$$

In mathematical terms, (14) simply states that the subgrid scale stress tensor is a functional of (i.e. is a quantity determined by) the global history of the mean velocity field.

Now, we will examine some of the subgrid scale stress models which have been utilized over the past decade. In virtually all of these subgrid scale stress models, the subgrid scale Reynolds stresses are modeled by using an eddy viscosity approach which is of the form

$$R_{kl} = -2\nu_f \bar{D}_{kl}. \quad (15)$$

†Since the Leonard stresses can be calculated directly, it is actually only necessary to provide closure models for $C$ and $R$. This fact has been made use of in the more recent subgrid scale stress models.
where
\[ \tilde{D}_{kl} = \frac{1}{2} (\partial \tilde{u}_i / \partial x_j + \partial \tilde{u}_j / \partial x_i) \]  
and \( v_T \) is the eddy (or turbulent) viscosity. Two different models have been used for the eddy viscosity \( v_T \). The most commonly used one is the Smagorinsky Model (1963) given by
\[ v_T = C_1 \Delta \sqrt{(\tilde{D}_{kl} \tilde{D}_{kl})^{1/2}}, \]  
where \( C_1 \) is a dimensionless constant. However, more recently, vorticity models have been constructed which are of the form (see Ferziger, 1977)
\[ v_T = C_2 \Delta \sqrt{(\partial \tilde{\omega}_x \partial \tilde{\omega}_x)^{1/2}}, \]  
where \( \tilde{\omega} = \nabla \times \tilde{u} \) is the mean vorticity vector and \( C_2 \) is a dimensionless constant.

A variety of approaches have been proposed for the modeling of the Leonard stresses and subgrid scale cross stresses. The earliest approach made use of the Reynolds averaging assumption (see Deardorff, 1970)
\[ L_{ai} + C_{ai} = 0. \]  
Equation (19) is approached in an asymptotic sense as the Reynolds numbers go to infinity. Subsequent to this work, Clark, Ferziger and Reynolds (1979) proposed the model
\[ L_{ai} + C_{ai} = (1/12) \Delta^2 (\partial \tilde{u}_i / \partial x_m)(\partial \tilde{u}_i / \partial x_m). \]  
However, more recent models have been constructed which, for improved accuracy, provide for the direct calculation of the Leonard stresses,
\[ L_{ai} = \tilde{u}_i \tilde{u}_i - \tilde{u}_k \tilde{u}_k \]  
by means of a convolution theorem (see Biringen and Reynolds, 1981). The first approach along these lines, which is still being used, consists of the construction of a modified eddy viscosity model of the
where the eddy viscosity is formulated using the Smagorinsky model (see Biringen and Reynolds, 1981 and Moin and Kim, 1982). The most recent such model is that proposed by Bardina, Ferziger, and Reynolds (1983) which is of the form

\[ C_{kl} + R_{kl} = -2v_f \tilde{D}_{kl} \]  

(22)

where \( C_{kl} \) is a dimensionless constant and the eddy viscosity \( v_f \) is usually formulated with the Smagorinsky Model (17). This model has been referred to as the linear combination model. In the next sections we will examine the consistency of these models with certain physical properties which can be derived as a rigorous consequence of the Navier–Stokes equations.

3. THE INVARIANCE AND DISSIPATIVE STRUCTURE OF SUBGRID SCALE STRESS MODELS

Since the Navier–Stokes equations as well as their filtered form (7) are invariant under the Galilean group of transformations, all subgrid scale stress models must exhibit the same invariance or they are physically incorrect. More specifically, Eq. (14) must transform in the form invariant manner

\[ \tau^{*}(x, t) = \tau^{*}(\bar{u}^{*}(x', t'; x, t)) \quad x' \in D, \quad t' \in (-\infty, t) \]  

(24)

under the Galilean group of transformations

\[ x^{*} = x + V t + B \]  

(25)

where \( V \) and \( B \) are constant vectors. In Speziale (1985a) it was demonstrated that the modified eddy viscosity model (22) used by Biringen and Reynolds (1981) and Moin and Kim (1982) is not Galilean invariant and, hence, is inconsistent with the basic physics of the problem which requires that the description of the turbulence be the same in all inertial frames of reference. This problem arises
since the cross stresses are not Galilean invariant while $R$ and $D$ are. Hence, Eq. (22) transforms as

$$C_{k l}^* + R_{k l}^* - V_k u_l^* - V_l u_k^* = -2v^* D_{k l}^*$$

under the Galilean group of transformations (see Speziale, 1985a) which is *not* form invariant since it depends explicitly on the translational velocity $V$ of the inertial frame of reference. Since (22) is not Galilean invariant it gives rise to equations of motion for the evolution of the large eddies which are similarly not invariant. This is unacceptable on physical grounds, and, hence, modified eddy viscosity models of the form (22) should be discarded.

Under the Galilean group of transformations, the linear combination model (23) transforms as

$$C_{k l} + R_{k l} - V_k u_l - V_l u_k = c_r (\bar{u}_k^* \bar{u}_l - \bar{u}_k \bar{u}_l^*) - c_r (V_k u_l^* + V_l u_k^*) - 2v^* D_{k l}$$

and is, thus, form invariant if and only if (see Speziale, 1985a)

$$c_r = 1.$$}

Interestingly enough, a value of $c_r = 1.1$ was arrived at by Bardina, Ferziger, and Reynolds (1983) by correlating with the results of direct numerical simulations. However, in light of this analysis, for any future calculations with the linear combination model the constant $c_r$ should be set equal to a value of 1. It should be noted at this point that all of the remaining subgrid scale stress models (i.e., Eqs. (15H20)) discussed in Section 2 are Galilean invariant.

Now, the transformation properties of subgrid scale stress models will be explored in non-inertial frames of reference which can undergo arbitrary time-dependent rotations and translations relative to an inertial framing. In mathematical terms, we will consider the invariance properties of subgrid scale stress models under the Euclidean group of transformations

$$x^* = Q(t)x + b(t), \quad t^* = t + a,$$
where

\[ QQ^T = Q^T Q = I, \quad |Q| = 1. \] (30)

In (29) and (30), \( b(t) \) is any time-dependent vector, \( Q(t) \) is any time-dependent proper orthogonal tensor, \( a \) is any constant, \( I \) is the unit tensor, the superscript \( T \) denotes the transpose, and \( |\cdot| \) denotes the determinant. Given that \( x \) is an inertial frame of reference, \( x^* \) given by (29) will represent an arbitrary non-inertial frame of reference. It is a simple matter to show that

\[ \dot{Q}_{km} Q_{lm} = e_{klm} \Omega_m, \quad \delta_k = -\Omega_k, \] (31)

where \( e \) is the permutation tensor and \( \Omega(t) \) and \( V(t) \) are, respectively, the angular velocity and translational velocity of the non-inertial frame of reference relative to an inertial framing.

Relative to an arbitrary non-inertial frame of reference, the Navier–Stokes equations take the form (cf. Batchelor, 1967)

\[ \rho [\partial u^*/\partial t^* + (u^* \cdot V^*) u^*] = -V^* P^* + \mu V^* u^* - \rho \Omega x^* - 2 \rho \Omega \times u^*, \] (32)

\[ V^* \cdot u^* = 0, \] (33)

where

\[ P^* = p^* - \frac{1}{2} \rho (\Omega \cdot \Omega)(x^* \cdot x^*) + \frac{1}{2} \rho (\Omega \cdot x^*)^2 + \rho V \cdot x^* \] (34)

is the modified pressure which includes the centrifugal and translational acceleration potentials of the non-inertial framing. The filtered form of the equations of motion (32) and (33) are as follows

\[ \rho [\partial \tilde{u}^*/\partial t^* + (\tilde{u}^* \cdot V^*) \tilde{u}^*] = -V^* \tilde{P}^* + \mu V^* \tilde{u}^* - \rho \tilde{\Omega} x^* \]

\[ - 2 \rho \Omega \times \tilde{u}^* - \rho V^* \cdot \tau^*, \] (35)

\[ V^* \cdot \tilde{u}^* = 0, \] (36)

where

\[ \tau_{kk}^* = \tilde{u}_{k*}^* \tilde{u}_{k*}^* - \tilde{u}_{k*}^* \bar{u}_{k*}^* + \bar{u}_{k*}^* \bar{u}_{k*}^* + \bar{u}_{k*}^* \bar{u}_{k*}^* + \bar{u}_{k*}^* \bar{u}_{k*}^*. \] (37)
In deriving (35), we have made use of the fact that \( \mathbf{x} = \mathbf{x}^* \) and, hence,

\[
\mathbf{x}^* = \mathbf{x}^*. \tag{38}
\]

This is true since we will restrict our attention to isotropic filter functions of the form (see Appendix A)

\[
G(\mathbf{x}' - \mathbf{x}, \Delta) = G(\mathbf{x}' - \mathbf{x}, \Delta), \tag{39}
\]

which, in an infinite flow domain, will be taken to be a Gaussian distribution (cf. Biringen and Reynolds, 1981) and, in a finite flow domain, will be taken to be of the form

\[
G(|\mathbf{x}' - \mathbf{x}|, \Delta) = \begin{cases} F(|\mathbf{x}' - \mathbf{x}|, \Delta), & |\mathbf{x}' - \mathbf{x}| \leq \Delta, \\ 0, & |\mathbf{x}' - \mathbf{x}| \geq \Delta, \end{cases} \tag{40}
\]

where \( F \) is an infinitely differentiable positive function which is normalized (integrates to 1 in the region \( |\mathbf{x}' - \mathbf{x}| \leq \Delta \)) and vanishes at \( |\mathbf{x}' - \mathbf{x}| = \Delta \). In this manner, the filtering process will not depend on the frame of reference, i.e.

\[
G(|\mathbf{x}' - \mathbf{x}|) = G(|\mathbf{x}'^* - \mathbf{x}^*|), \tag{41}
\]

and, furthermore, for any summable function \( \phi \), the theory of distributions (cf. Arfken, 1970) guarantees that

\[
\lim_{\Delta \to 0} \int_D G(|\mathbf{x}' - \mathbf{x}|)\phi(\mathbf{x}')\,d^3\mathbf{x}' = \int_D \delta(\mathbf{x}' - \mathbf{x})\phi(\mathbf{x}')\,d^3\mathbf{x}' = \phi(\mathbf{x}). \tag{42}
\]

Hence, \( G \) will always constitute a Dirac delta sequence which is required on physical grounds (i.e. as the mesh size goes to zero, all the scales are resolved and there is no filtering).

Equations (35) and (36) can also be obtained by substituting the Euclidean transformations of \( \mathbf{u} \) and \( \mathbf{u}' \) into (7) and (8). Since

\[
\mathbf{u} = \mathbf{u}^* + \Omega \times \mathbf{x}^* + \mathbf{V}, \tag{43}
\]

it is clear that these transformations are as follows:

\[
\mathbf{u}^* = \mathbf{u}^* + \Omega \times \mathbf{x}^* + \mathbf{V}, \quad \mathbf{u}' = \mathbf{u}^*. \tag{44}, (45)}
From (45), it is obvious that the fluctuating velocity is a frame-indifferent vector and, hence, two observers whose motions differ by an arbitrary time-dependent rotation and translation would measure the same fluctuating velocity for a given turbulent flow. As a direct consequence of (45), it is straightforward to show that the subgrid scale Reynolds stress tensor $R$ is a frame-indifferent tensor, i.e. $R$ transforms as

$$R^* = R$$ (46)

under the Euclidean group of transformations (29) and is thus independent of the observer. However, it is interesting to note that the remaining subgrid scale stresses are not frame-indifferent, i.e. under a Euclidean transformation

$$L^* + C^* = L + C - (\Omega \times x^*)u^* - u^*(\Omega \times x^*) + (\Omega \times x^*)v^* + v^*(\Omega \times x^*)$$

$$+ (\Omega \times x^*)(\Omega \times x^*) - (\Omega \times x^*)(\Omega \times x^*)$$ (47)

and, hence, $L+C$ depends on the motion of the frame of reference. Nevertheless, it can be shown that (see Appendix B)

$$\nabla^* \cdot (L^* + C^*) = \nabla \cdot (L + C).$$ (48)

Consequently, the divergence of the subgrid scale stress tensor is a frame-indifferent vector, i.e.

$$\nabla^* \cdot \tau^* = \nabla \cdot \tau.$$ (49)

The frame-dependence of $L+C$ is not of any serious consequence since only its divergence (which is frame-indifferent) enters into the calculation of the large eddies.

Since $R$ and $\nabla \cdot (L+C)$ are frame-indifferent tensors we can now address the question as to whether or not the principle of material frame-indifference (see Truesdell and Noll, 1965) can be invoked in the modeling of these terms. The principle of material frame-indifference for the subgrid scale Reynolds stress tensor $R$ would require that the models for this term be form invariant under a change of frame. In mathematical terms, $R$ would have to transform
in the invariant manner

\[ R^*(x, t) = R[\bar{u}^*(x', t'); x, t], \quad x' \in D, \; t' \in (-\infty, t) \]  

(50)

under the Euclidean group of transformations (29). The physical consequence of (50) is as follows: if the same mean velocity history is produced in an inertial frame of reference and in an arbitrary non-inertial frame of reference, the Reynolds subgrid scale stress tensors would be identical in the two cases. Consequently, \( \mathbf{R} \) would be unaffected by the state of rotation of the mean velocity. In order to determine the validity of such a hypothesis, it is necessary to examine the equations of motion for the fluctuating velocity in a non-inertial frame of reference. These equations, which are obtained by subtracting (35) and (36) from (32) and (33), are as follows:

\[ \rho [\partial u'/\partial t^* + (u^* \cdot \nabla^*) u^*] = -\rho (u'^* \cdot \nabla^*) u'^* \\
-\rho (u'^* \cdot \nabla^*) \bar{u}^* - \nabla^* \mathbf{P}^* \\
+ \mu \nabla^* u'^* + \rho \nabla^* \cdot \mathbf{t}^* - 2\rho \Omega \times \mathbf{u}^*, \quad (51) \]

\[ \nabla^* \cdot u'^* = 0. \quad (52) \]

As a result of (51), it is clear that if we force the same mean velocity in the non-inertial framing we will not necessarily obtain the same fluctuating velocity because of the presence of the Coriolis term on the right-hand side of (51). In mathematical terms, if

\[ \bar{u}^* = \bar{u}, \quad (53) \]

then, in general

\[ u'^* \neq u'. \quad (54) \]

Since \( \mathbf{R}^* \) is constructed directly from \( u'^* \), there is then reason to doubt the general validity of (50). However, it is clear that subgrid scale stress models should be form invariant under the extended Galilean group of transformations

\[ x^* = x + b(t) \]
since the effects of translational accelerations of the frame of reference are subtracted out of (51) leaving the fluctuating velocity unaffected. The extended Galilean group represents a class of non-inertial frames of reference which can undergo an arbitrary time-dependent translation relative to an inertial framing. It is a simple matter to show that the subgrid scale stress models discussed in the previous section which are Galilean invariant are also invariant under the extended Galilean group. Hence, this constraint is satisfied by such models.

The limit of two-dimensional turbulence can be approached in a rapidly rotating framework far from solid boundaries as a result of the Taylor–Proudman theorem (cf. Greenspan, 1968). For a two dimensional turbulence where

\[ u^* = u^*(x_1^*, x_2^*, \tau^*), \quad \bar{u}^* = \bar{u}^*(x_1^*, x_2^*, \tau^*), \]

there exists a fluctuating stream function \( \psi^*(x_1^*, x_2^*, \tau^*) \) so that

\[ u^* = \nabla^* \psi^* \times e_1^* + u_1^* e_1^*, \]

where \( \Omega = \Omega e_1^* \) and \( e_1^* \) is a unit vector in the \( x_1^* \) direction. As a direct consequence of (56), the Coriolis acceleration is derivable from a scalar potential, i.e.

\[ 2\rho \Omega \times u^* = \nabla^* (2\rho \Omega \psi^*). \]

The substitution of (57) into (51) yields the fluctuating momentum equation

\[ \rho \left[ \partial u^*/\partial \tau^* + (\bar{u}^* \cdot \nabla^*) u^* \right] = -\rho (u^* \cdot \nabla^*) u^* - \rho (u^* \cdot \nabla^*) \bar{u}^* - \nabla^* (P^* + 2\rho \Omega \psi^*) + \mu \nabla^2 u^* + \rho \nabla^* \tau^*. \]

Hence, if we set

\[ \bar{u}^* = \bar{u} \]

in (58), then

\[ u^* = u', \quad R^* = R, \]

\[ P^* = P' - 2\rho \Omega \psi^*, \]
SIMULATION OF ROTATING TURBULENCE

and it follows that for a two-dimensional turbulence the evolution of
a velocity fluctuation is unaffected by the state of rotation of the
mean velocity. Consequently, for such a turbulence the principle of
material frame-indifference (50) is a rigorous consequence of the
Navier–Stokes equations. Furthermore, it can also be shown that
provided the disparity in the time scales of the subgrid scale and
mean motions is large, material frame-indifference would be approxi-
mately valid for more general turbulent flows. This can easily be
seen if we non-dimensionalize the fluctuating momentum equation
using the following terms

\[ x^+ = x^*/l_0, \quad t^+ = t^*/t_0, \quad u'^+ = u'^*/l_0, \]

\[ \tilde{u}^+ = \tilde{u}^* T_0/L_0, \quad P'^+ = P^* t_0^2/l_0^2 \rho, \quad \Omega^+ = \Omega T_0, \]

where \( t_0 \) is the time scale of the subgrid scale flow, \( l_0 \) is the length
scale of the subgrid scale flow, \( T_0 \) is the time scale of the mean flow,
and \( L_0 \) is the length scale of the mean flow.† The use of (62) and (63)
in (51) yields the dimensionless fluctuating momentum equation

\[ \partial u'^+ / \partial t^+ + \tilde{u}^+ \cdot \nabla u'^+ = - (u'^+ \cdot \nabla^+) u'^+ - (u'^+ \cdot \nabla^+) \tilde{u}^+ - \nabla^+ P'^+ + \nabla^+ \cdot \tau^+ - 2(t_0/T_0) \Omega^+ \times \tilde{u}^+, \]

where the Reynolds number

\[ \text{Re} = \rho v_0 l_0 / \mu, \quad v_0 = l_0 / t_0, \]

and, for simplicity, we have assumed that \( l_0 / L_0 \approx t_0 / T_0 \). Provided that

\[ t_0 / T_0 \ll 1, \]

(i.e. provided that the disparity in the time scales of the subgrid scale
and mean motions is extremely large), the effect of the Coriolis
acceleration on the right-hand side of (64) will be negligible and
material frame-indifference will be valid in a strong approximate

†This choice of scales guarantees that the dimensionless field variables in (62) and
(63) are normalized to one.
sense for more general turbulent flows. For most applications,

\[
\frac{t_0}{T_0} \approx \frac{\Delta}{L_0} \approx 10^{-2}
\]  

(67)

and, hence, material frame-indifference for general turbulent flows would only appear to be valid in a weak approximate sense. However, it should be noted that all of the currently popular Reynolds subgrid scale stress models discussed in Section 2 constitute local theories whose very existence is predicated on the assumption that (cf., Chapman and Cowling, 1953)

\[
\frac{t_0}{T_0} \ll 1
\]  

(68)

Considering this fact along with the just established result that the limiting case of two-dimensional turbulence is frame-indifferent,† it would appear to be logically inconsistent to formulate local subgrid scale stress models that are frame-dependent.

It should be noted at this point that the results proven above also apply to subgrid scale stress models for \( \nabla \cdot (L+C) \). To be more specific, as a direct consequence of (58)–(66), material frame-indifference must be satisfied identically for the limiting case of two-dimensional turbulence and should be generally valid in an approximate sense for any subgrid scale stress models constructed for \( \nabla \cdot (L+C) \).

Now, we will examine the effect of rotations on the turbulence dissipation. Bardina, Ferziger, and Reynolds (1983) discovered in their large-eddy simulation of rotating turbulent flows that a significant increase in the rotation rate is accompanied by a substantial reduction in the energy dissipation rate of the turbulence. They argued that the inertial waves generated by the rotation destroy the phase coherence needed to cascade energy from the large eddies to the small eddies which leads to a reduction in the dissipation. While this argument is certainly correct, a more direct physical explanation

†Hence, as a result of the Taylor–Proudman theorem, one need only be concerned with the case when \( \Omega^* < 10 \); but, for this case, (68) guarantees that the effects of the Coriolis term are small.
of this phenomenon will be presented herein. A significant increase in
the rotation rate of the fluid leads to a Taylor–Proudman reorganiza-
tion of flow. In fact, in the limit of infinite rotation rates the flow
becomes two-dimensional as discussed earlier (see Greenspan, 1968
and Speziale, 1985b). It is a well established fact that in a two-
dimensional turbulence it becomes increasingly difficult to cascade
energy from the large eddies to the small eddies as a result of
spectral blocking (see Fjortoft, 1953). For infinite Reynolds numbers,
the energy dissipation rate vanishes and the kinetic energy becomes
a conservative quantity (see Batchelor, 1969). Hence, at rapid
rotation rates, a substantial reduction in the dissipation rate of the
turbulent kinetic energy is to be expected on physical grounds. The
small eddies then no longer serve primarily as a source for dissi-
pation and non-dissipative momentum exchanges between the large
and small scales of the turbulence are enhanced. In the next section,
it will be demonstrated that these results place severe constraints on
the allowable form of subgrid scale stress models and can, thus, serve
as a powerful modeling tool.

4. COMPARISON OF SUBGRID SCALE STRESS MODELS
FOR ROTATING FLOWS

By utilizing the results derived in the previous section, it will first be
demonstrated that vorticity subgrid scale stress models are funda-
mentally inconsistent with the Navier–Stokes equations for rotating
flows. Relative to an arbitrary non-inertial frame of reference the
vorticity subgrid scale stress model defined by (15) and (18) takes the
form

\[ R_{*i} = -2C_2 \Delta^2 \tilde{\omega}^* + 2\Omega \tilde{D}_{*i} \]  \hspace{1cm} (69)

and is thus frame-dependent because of the presence of a term that
contains \( \Omega \). Equation (69) is obtained by using (46) and the
transformations

\[ \tilde{\omega} = \tilde{\omega}^* + 2\Omega, \quad \tilde{D}^* = \tilde{D} \]  \hspace{1cm} (70)

under a change of frame which can be derived by differentiating (43).
For a two-dimensional turbulence, the inertial term in (69) is still present and, consequently, vorticity subgrid scale stress models are inconsistent with the constraint that material frame-indifference be satisfied for such a turbulence—a constraint which is a rigorous consequence of the Navier–Stokes equations as proven above. This is a rather serious inconsistency since the limit of two-dimensional turbulence constitutes a real physical limit which, in principle, can be approached by any turbulence in a rapidly rotating framework that is sufficiently far from solid boundaries. Furthermore, the vorticity subgrid scale stress model (69) is inconsistent with the dissipative properties of the Navier–Stokes equations that were discussed in the previous section. This can be illustrated by the following simple example: a homogeneous turbulence, at extremely high Reynolds numbers, in a rotating framework. For this case, the dissipation rate of the filtered turbulent kinetic energy can be approximated by (see Bardina, Ferziger and Reynolds, 1983).

$$\varepsilon_f \approx \langle -\tau_{ij} \hat{D}_{ij} \rangle,$$

(71)

where $\langle \cdot \rangle$ denotes a spatial average. By utilizing (71), it is quite clear that the vorticity subgrid scale stress model (69) gives rise to a dissipation rate which can be approximated by

$$\varepsilon_f \approx 2C_2 \Delta^2 \langle \hat{\omega}^* + 2\Omega |\hat{D}_{ij}^*| \hat{D}_{ij}^* \rangle,$$

(72)

and, hence, $\varepsilon_f \to \infty$ as $\Omega \to \infty$. But according to the results derived earlier, $\varepsilon_f \to 0$ as a result of the spectral blocking tendency of a two-dimensional turbulence discussed in the last section. It is thus clear that vorticity subgrid scale stress models are fundamentally inconsistent with the Navier–Stokes equations for rotating flows. Bardina, Ferziger and Reynolds (1983) and Ferziger (1977) have concluded that there is no substantial difference between vorticity models and the Smagorinsky model. However, in their numerical analyses, the inertial term containing $\Omega$ in (69) was neglected which, of course, cannot be justified on physical grounds. Had the authors taken this term into account, the results of this paper strongly indicate that they would have come to a different conclusion, namely that vorticity models are physically inconsistent.

Now, we will consider the Smagorinsky model applied with the
Reynolds averaging assumption (cf. Deardorff, 1970). In an arbitrary non-inertial frame of reference, this model takes the form

\[ R^* = -2C_1 \Delta^2 (\bar{D}_{max} \Delta^* \bar{D}^*)^{1/2} \Delta \]

(73)

\[ \nabla^* \cdot (L^* + C^*) = 0, \]

(74)

which is obtained by substituting (46), (48) and (70) into (15) and (19) where \( \nu_T \) is given by (17). Obviously this model is frame-indifferent since no inertial terms appear, and hence this model is properly invariant. However, there are some problems with this model which make its general applicability to rotating flows questionable. Calculations which have been done in recent years (cf. Clark, Ferziger and Reynolds, 1979) cast serious doubts on the validity of the Reynolds averaging assumption. Furthermore, the subgrid scale stresses serve exclusively as a source for dissipation in this model since

\[ \epsilon_f \approx 2C_1 \Delta^2 \langle (\bar{D}_{max} \Delta^*)^{3/2} \rangle \geq 0. \]

(75)

However, for the limit of two-dimensional turbulence (i.e. for a turbulence in a rapidly rotating framework) the small eddies do not play a purely dissipative role; non-dissipative momentum exchanges between the large and small scales are enhanced significantly.

The model of Clark, Ferziger and Reynolds (1979) when used with the Smagorinsky model is also frame-indifferent. More specifically, relative to an arbitrary non-inertial frame of reference, this model takes the form

\[ R^*_m = -2C_1 \Delta^2 (\bar{D}_{max} \Delta^*)^{1/2} \Delta^*_m, \]

(76)

\[ \frac{\partial}{\partial x^*_m} (L^*_m + C^*_m) = \frac{1}{12} \Delta^2 \frac{\partial}{\partial x^*_m} \left( \frac{\partial \bar{u}^*_m}{\partial x^*_m} \frac{\partial \bar{u}^*_m}{\partial x^*_m} \right), \]

(77)

and is thus form invariant under a change of frame. Equation (77) is obtained by utilizing (43) and (48) in (20). Hence, this model is properly invariant. Furthermore, as a result of the fact that in a non-inertial frame of reference the modeled version of the subgrid scale
stress tensor takes the form

\[ \tau_{kl}^* = -2C_1 \Delta^2 (\delta_{mn} \delta_{mn})^{1/2} \delta_{kl}^* + (1/12) \Delta^2 (\partial_{kk}^* / \partial x_m^* + e_{kmn}^* \Omega_n) (\partial_{kk}^* / \partial x_m^* + e_{kmn}^* \Omega_n), \]  

(78)

it has a viscoelastic character that becomes more pronounced as the rotation rate of the framing is increased. To be more specific, the quadratic term in \( V^* \bar{\Omega}^* \) in (78) is a viscoelastic term (see Truesdell and Noll, 1965) which can store energy rather than just dissipate it. Consequently, as the rotation rate of the framing is increased, the subgrid scale stresses will have less of a dissipative character consistent with the result derived in the previous section. The subgrid scale stress model of Clark, Ferziger and Reynolds (1979) thus has the correct behavior for the description of rotating turbulent flows.

Finally, it will be demonstrated that the linear combination model of Bardina, Ferziger and Reynolds (1983) also satisfies the constraints derived in the previous section provided that \( c_s = 1 \). For \( c_s = 1 \), the Bardina, Ferziger and Reynolds (1983) model takes the form

\[ R_{kl} = -2C_1 \Delta^2 (\bar{D}_{mn}^* \delta_{mn}^*)^{1/2} \bar{D}_{kl}^*, \]  

(79)

\[ L_{kl} + C_{kl} = \bar{u}_{kl} - \bar{u}_{kl}, \]  

(80)

in an inertial frame of reference. In a non-inertial frame of reference, this model takes the form

\[ R^* = -2C_1 \Delta^2 (\bar{D}_{mn}^* \delta_{mn}^*)^{1/2} \bar{D}^*, \]  

(81)

\[ V^* (L^* + C^*) = V^* (\bar{u}^* \bar{u}^* - \bar{u}^* \bar{u}^*), \]  

(82)

and, hence, it is frame-indifferent since no inertial terms appear. Equation (82) is obtained by taking the divergence of the non-inertial form of the right-hand side of (80) given by

\[ \bar{u}^* \bar{u}^* - \bar{u}^* \bar{u}^* = \bar{u}^* \bar{u}^* - \bar{u}^* \bar{u}^* + \bar{u}^* (\Omega \times x^*) + (\Omega \times x^*) \bar{u}^* + (\Omega \times x^*) (\Omega \times x^*) \]

\[ - (\Omega \times x^*) (\Omega \times x^*) - \bar{u}^* (\Omega \times x^*) - (\Omega \times x^*) \bar{u}^*, \]  

(83)
and making use of the results in Appendix B along with (48). It is clear from (83) that the non-dissipative part \( L + C \) of the linear combination model is enhanced in a rotating framework as a result of the presence of the inertial terms [this observation is consistent with the calculations of Bardina, Ferziger and Reynolds (1983) which yielded a reduction in the dissipation rate for rotating flows]. Hence, the linear combination model of Bardina, Ferziger and Reynolds (1983) is consistent with the constraints derived herein and is thus quite suitable for the description of rotating flows. In addition, it should be noted that this linear combination model has certain advantages over the Clark, Ferziger and Reynolds (1979) model in that the Leonard stresses are calculated directly and the model for the cross stresses is easier to implement because of the absence of spatial gradients. These features, of course, lead to improved accuracy and make the linear combination model the best model that is currently available for the description of rotating flows.

5. CONCLUSION

A theoretical investigation has been conducted on the constraints that subgrid scale stress models must be subject to if they are to be suitable for the large-eddy simulation of rotating turbulent flows. It was proven that such models must be form invariant under the extended Galilean group and must be frame-indifferent in the limit of two-dimensional turbulence which can be approached in a rapidly rotating framework (sufficiently far from solid boundaries) as a direct consequence of the Taylor–Proudman theorem. Furthermore, it was proven that provided the disparity in the time scales of the subgrid scale and mean motions is large, frame-indifference is valid in a strong approximate sense for more general turbulent flows. Since most existing subgrid scale stress models are local which requires that such a disparity in scales exists, it was argued that material frame-indifference can be invoked as an axiom for such models. Finally, it was shown that a significant increase in the rotation rate of the fluid must be accompanied by a substantial reduction in the turbulence dissipation rate as a result of the induced Taylor–Proudman reorganization with its associated spectral blocking tendency. Vorticity subgrid scale stress models along with some other
commonly used models were found to be in serious violation of these constraints and, consequently, they are unsuitable for the description of rotating flows. Only the model of Clark, Ferziger and Reynolds (1979) and the modified linear combination model of Bardina, Ferziger and Reynolds (1983) are consistent with these constraints with the latter model preferred because of its more accurate treatment of the Leonard stresses.

The results of this paper clearly demonstrate that the modified linear combination model of Bardina, Ferziger and Reynolds (1983) is significantly the best existing subgrid scale stress model for simulating rotating turbulent flows. This model is quite sound from a fundamental mechanics standpoint and should be of considerable value in future large-eddy simulations. However, there is still more research needed on subgrid scale stress modeling. More research is needed on the quantitative effect of rotations on the turbulence dissipation in order to refine existing models (the results presented herein were, for the most part, qualitative in this regard). Future research is also needed to examine the importance of history dependent effects which are neglected in this model. Since such effects are known to be extremely important in the description of the large eddies, their effect might be non-negligible in the formulation of subgrid scale stress models. It would appear that the refinement of subgrid scale stress models will be a continuing effort which will be guided by the specific numerical results obtained from large-eddy simulations and by a closer examination of the fundamental physics of the problem.

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References


SIMULATION OF ROTATING TURBULENCE


Appendix A

By definition, \( \mathbf{x} \) is given by

\[
\mathbf{x} = \int_\mathcal{D} G(|\mathbf{x}' - \mathbf{x}|) \mathbf{x}' \, d^3\mathbf{x}'.
\]

(A.1)

If the change of variable

\[
\mathbf{x}' - \mathbf{x} = \xi, \quad |\mathbf{x}' - \mathbf{x}| = \xi
\]

(A.2)

is introduced, (A.1) takes the form

\[
\mathbf{\hat{x}} = \int_\mathcal{D} G(\xi)(\mathbf{x} + \xi) \, d^3\xi
\]

\[
= \mathbf{x} \int_\mathcal{D} G(\xi) \, d^3\xi + \int_\mathcal{D} G(\xi)\xi \, d^3\xi.
\]

(A.3)
However, it is clear that
\[
\int_{b} G(\xi) \, d^3 \xi = \int_{b} G(|x' - x|) \, d^3 x' = 1. \quad (A.4)
\]
Furthermore, since \( G(\xi) = 0 \) for \( \xi \geq \Delta \) it follows that
\[
\int_{b} G(\xi) \, d^3 \xi \int_{|\xi| \leq 1} G(\xi) \, d^3 \xi = 0, \quad (A.5)
\]
because \( G(\xi) \) is an even function while \( \xi \) is an odd function. Hence, it follows from (A.3) that
\[
\bar{x} = x. \quad (A.6)
\]
Furthermore, under a change of frame
\[
x^* = Q(t)x + b(t), \quad (A.7)
\]
\( G \) and \( d^3 x' \) transform as
\[
G(|x^* - x|) = G(|x' - x|), \quad d^3 x^* = d^3 x'. \quad (A.8)
\]
Thus, from (A.1) and (A.6) it is clear that
\[
\bar{x} = x^*, \quad (A.9)
\]
which completes the proof.

Appendix B

In order to establish the identity
\[
V^* \cdot (L^* + C^*) = V \cdot (L + C), \quad (B.1)
\]
it is only necessary to show that
\[
V^* \cdot \left[ - (\Omega \times x^*)u^* - u^*(\Omega \times x^*) + (\Omega \times x^*)\bar{u}^* + \bar{u}^*(\Omega \times x^*) \right. \\
+ (\Omega \times x^*)(\Omega \times x^*) - (\Omega \times x^*)(\Omega \times x^*) \right] = 0 \quad (B.2)
\]
as a result of (47). The left-hand side of (B.2), in indicial notation, takes the form

\[
(\partial/\partial x_i^*) [- \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} u_i^* - \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} u_i^* + \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} u_i^* \\
+ \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^* + \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^* - \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^*]
\]

\[
= - \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^* - \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^* + \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^*
\]

\[
+ \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^* + \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^* - \varepsilon_{km} \Omega_{m\cdot n} x^*_{n} \delta_{k}^*
\]

\[
= \varepsilon_{km} \Omega_{m\cdot n} (x^*_{n} \delta_{k}^* - x^*_{n} \delta_{k}^* - x^*_{n} \delta_{k}^* - x^*_{n} \delta_{k}^*),
\]

(B.3)

since \( x^* = x^* \). However,

\[
x^* \delta u_i^* / \delta x_i^* - x^* \delta u_i^* / \delta x_i^*
\]

\[
= \int_D G(|x^* - x^*|)(x^* - x^*) \delta u_i^*(x^*) / \delta x_i^* d^3 x^*
\]

\[
= - \int_D \partial H(|x^* - x^*|) / \partial x_i^* \delta u_i^*(x^*) d^3 x^*,
\]

(B.4)

where \( H \) is an integral of \( G \), i.e.

\[
\xi^{-1} dH(\xi)/d\xi = G(\xi).
\]

(B.5)

The integration of (B.4) by parts yields the result

\[
x^* \delta u_i^* / \delta x_i^* - x^* \delta u_i^* / \delta x_i^*
\]

\[
= \int_D H(|x^* - x^*|) \partial^2 u_i^*(x^*) / \partial x_i^* d^3 x^*
\]

\[
- \int_D H(|x^* - x^*|) \partial u_i^*(x^*) / \partial x_i^* dS^*,
\]

(B.6)

where \( \partial D \) is the boundary surface of \( D \). Since, as a result of (B.5), \( H \)
is only defined to within an additive constant we can select \( H \) so that it vanishes on \( \partial D \) (i.e. on the surface \(|x^* - x|=\Delta\)). This eliminates the surface integral in (B.6). Hence, we have

\[
e_{lmn} \Omega_m (x^*_m \partial u^*_l / \partial x^*_l - x^*_n \partial u^*_n / \partial x^*_n),
\]

\[
= e_{lmn} \Omega_m \left[ H(x^* - x^*) \left[ \partial^2 u^*_k / \partial x^*_k \partial x^*_n \right] d^3 x^* \right] = 0,
\]

(B.7)

since \( \varepsilon \) and \( \nabla^* \nabla^* u^* \) are, respectively, antisymmetric and symmetric tensors in the indices \( l \) and \( n \). Thus, (B.2) is valid which then immediately yields (B.1): the frame-indifference of \( \nabla \cdot (L + C) \).