FORMULATION OF DRUCKER-PRAGER CAP MODEL

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ABSTRACT: The Drucker-Prager cap and similar models for the constitutive behavior of geotechnical materials are widely used in finite element stress analysis. They are multisurface plasticity models, used most frequently with an associated flow rule. The cap may harden or soften, and is coupled to the Drucker-Prager yield surface. As a result of this coupling, plastic deformation in pure shear is possible, after some plastic volume change, for any state of stress on the Drucker-Prager surface. This suggests that for full coupling the constitutive equations for the model can be found consistently; however, the model exhibits unstable behavior under certain conditions. To suppress this instability, some modification of the coupling must be made. Two examples of such modifications which appear in the literature are given; each leads to an inconsistent formulation. Numerical examples are used to illustrate differences and consequences arising from the different assumptions.

INTRODUCTION

In recent years a number of mathematical models of the constitutive behavior of granular or frictional materials have been proposed. These models originate in the fields of soil and rock mechanics, and in describing the mechanical behavior of concrete, and involve curve fitting, variable moduli, plastic and viscoplastic theories, fracturing theories and endochronic theory [see Nelson (11)]. In this group of material models, one of the most basic and most commonly implemented models in finite element programs is the plasticity model generally referred to as the Drucker-Prager yield condition with a cap.

The Drucker-Prager model (5) is elastic, perfectly plastic, with a yield surface that depends on hydrostatic pressure (in fact a cone in the principal stress space) and an associated flow rule. The primary shortcomings of the model are that it predicts plastic dilatancy that greatly exceeds what is observed experimentally, and that the behavior in hydrostatic compression is poorly represented. To overcome these deficiencies, Drucker, Gibson and Henkel (7) introduced a second yield function which hardens and, in the case of a soil, softens; this is the cap, so called because it closes the cone in the principal stress space. The shape of the cap in the principal stress space can be chosen in various ways; models developed by Sandler, et al. (3,13,14,15) use an elliptically shaped cap, whereas Bathe, et al. (1) allow only for a plane cap.

The constitutive equations for the cap describe behavior in hydrostatic compression, with hardening occurring when plastic deformation takes place. If, however, the Drucker-Prager cone and the cap are coupled,
through the plastic volume strain, the cap softens when plastic volume strain occurs on the cone. When the cap-cone vertex overtakes the stress point, plastic deformation in pure shear becomes possible. The introduction of the cap thus overcomes, to some extent, the principal difficulties in the Drucker-Prager model.

Our major concern is behavior when yielding occurs simultaneously on the Drucker-Prager cone and the cap. The yield surfaces are coupled, in the sense that the cap position depends on the total plastic volume strain produced on the Drucker-Prager and cap surfaces, among other parameters. The functional form of the yield surfaces, with full coupling and the assumption of an associated flow rule, is sufficient to permit the complete behavior during simultaneous yielding to be derived. However, full coupling is not assumed in the models of Sandler, et al. (3,13,14,15) and Bathe, et al. (1). This is in order to suppress an instability [in the sense that the stability postulate of Drucker (4,6) is not satisfied] that occurs in certain ranges of behavior. Sandler, et al. (3,13,14,15) chose a limited form of coupling, whereas Bathe, et al. (3,13,14,15) chose a limited form of coupling, whereas Bathe, et al. (1) impose additional assumptions on the plastic strain rate vector. Lade (8,9) and Desai, et al. (2), among others, make modifications of the same type.

A consistent treatment of coupled yield surfaces has been set out by Maier (10). In the following sections we apply this process to a fully coupled model, looking only at the case where both the Drucker-Prager and cap surfaces are active. A particular form of the failure surface and the cap are chosen for this illustration, but the general conclusions are not limited to this choice. Stress rates are written in terms of strain rates for all regimes in the shear strain rate and volume strain rate space. Using this framework, we consider the models of Sandler and Rubin (15) and Bathe, et al. (1) (chosen because full details are given in the respective papers) and show that they are not fully consistent, for different reasons.

**Structure of Constitutive Equations**

Plasticity models provide inviscid relations between stress rate \( \dot{\sigma}_{ij} \) and strain rate \( \dot{\epsilon}_{ij} \). We assume that the total strain rate \( \dot{\epsilon}_{ij} \) can be written as the sum of an elastic and a plastic component:

\[
\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p \tag{1}
\]

and that the elastic behavior is isotropic. The elastic equations are written as

\[
\begin{align*}
\dot{\epsilon}_{kk}^e &= \frac{1}{3K} \dot{\sigma}_{kk} ; \\
\dot{\epsilon}_{ij}^e &= \frac{1}{2G} \dot{\sigma}_{ij} 
\end{align*} \tag{2}
\]

in which \( K \) and \( G \) = bulk and shear moduli, respectively; and \( \dot{\epsilon}_{ij} \) and \( \dot{\sigma}_{ij} \) = the deviatoric components of \( \dot{\epsilon}_{ij} \) and \( \dot{\sigma}_{ij} \).

The plastic strain rate \( \dot{\epsilon}_{ij}^p \) is given as the sum of contributions from the associated flow rate for \( n \) active yield surfaces:
\[ \dot{e}_{ij}^p = \lambda_a \frac{\partial F_a}{\partial \sigma_{ij}} \] .................................................. (3)

in which \( \alpha = 1, 2, \ldots, n \) and the summation rule applies. \( F_a \) = yield functions, and \( \lambda_a \) = non-negative multipliers, with

\[ \lambda_a \geq 0 \quad \text{if} \quad F_a = 0 \quad \text{and} \quad \dot{F}_a = 0; \]

\[ \lambda_a = 0 \quad \text{if} \quad F_a = 0 \quad \text{and} \quad \dot{F}_a < 0 \quad \text{or} \quad F_a < 0 \] ................. (4)

In the Drucker-Prager cap model the yield surfaces are assumed to depend on the first and second invariants of the stress tensor. For our purposes, we shall choose these invariants as the mean hydrostatic tension \( \sigma_m \) and an effective shear stress \( s \), where

\[ \sigma_m = \frac{1}{3} \sigma_{kk}; \quad s = \sqrt{\frac{1}{2} s_{ij} s_{ij}} \] .................................................. (5)

Eq. 3 thus becomes

\[ \dot{e}_{ij}^p = \lambda_a \left( \frac{\partial F_a}{\partial \sigma_m} \frac{\partial \sigma_m}{\partial \sigma_{ij}} + \frac{\partial F_a}{\partial s} \frac{\partial s}{\partial \sigma_{ij}} \right) \] .................................................. (6)

By standard manipulations, we see that

\[ \frac{\partial \sigma_m}{\partial \sigma_{ij}} = \frac{1}{3} \delta_{ij}; \quad \frac{\partial s}{\partial \sigma_{ij}} = \frac{1}{2s} s_{ij} \] .................................................. (7)

From these results, it follows that the plastic volume strain rate is

\[ \dot{e}_{kk} = \lambda_a \frac{\partial F_a}{\partial \sigma_m} \] .................................................. (8a)

and that the deviatoric plastic strain rate is

\[ \dot{e}_{ij}^p = \frac{\lambda_a}{2s} \frac{\partial F_a}{\partial s} s_{ij} \] .................................................. (8b)

Initially, when basic ideas will be developed, it is convenient to simplify these equations. In particular, it is convenient to sketch the yield surfaces and the plastic constitutive relations in a two-dimensional space of the invariants \( \sigma_m \) and \( s \). To be able to do this, we must define effective strain rate quantities which are conjugate to the stress invariants. The first of these is simple to define, and is the volume strain rate which we will now denote by \( \dot{e}_v \):

\[ \dot{e}_v = \dot{e}_{kk} \] .................................................. (9)

The effective shear strain rate we shall define as

\[ \dot{e} = \frac{1}{s} s_{ij} \dot{e}_{ij} \] .................................................. (10)

This definition gives a scalar strain rate of degree zero in the stress components, and

\[ s\dot{e} = s_{ij} \dot{e}_{ij} \] .................................................. (11)
We may break $\dot{\varepsilon}$ into elastic and plastic components $\dot{\varepsilon}^e$, $\dot{\varepsilon}^p$ without difficulty. The shear stress rate $\dot{s}$ is obtained by differentiating Eq. 5, and is

$$\dot{s} = \frac{1}{2s_{ij} s_{ij}} \dot{s}.$$  \hspace{1cm} (12)

Using these definitions, we may now cast the constitutive equations in a very simple form. We have

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p; \quad \dot{\varepsilon}_p = \dot{\varepsilon}_p^e + \dot{\varepsilon}_p^p.$$  \hspace{1cm} (13)

with the elastic relations, from Eq. 2, given by

$$\dot{\varepsilon}_e^e = \frac{\sigma_m}{K}; \quad \dot{\varepsilon}_e^e = \frac{s}{G}.$$  \hspace{1cm} (14a)

and the plastic relations, from Eq. 8, given by

$$\dot{\varepsilon}_p^p = \lambda \frac{\partial F_1}{\partial \sigma_m}; \quad \dot{\varepsilon}_p^p = \lambda \frac{\partial F_1}{\partial s}.$$  \hspace{1cm} (14b)

Stability in the sense of Drucker (4, 6) is defined in terms of the second-order work. Appendix II shows that

$$\dot{s} \dot{\varepsilon} \leq s_{ij} \dot{s}_{ij}; \quad \dot{\sigma}_m \dot{\varepsilon}_m = \frac{1}{3} \dot{\sigma}_{kk} \dot{\varepsilon}_{kk}.$$  \hspace{1cm} (15a)

and thus it follows that if

$$\dot{s} \dot{\varepsilon} + \dot{\sigma}_m \dot{\varepsilon}_m \geq 0.$$  \hspace{1cm} (15b)

then

$$\dot{s}_{ij} \dot{\varepsilon}_{ij} \geq 0.$$  \hspace{1cm} (15c)

If, however, the sign of $(\dot{s} \dot{\varepsilon} + \dot{\sigma}_m \dot{\varepsilon}_m)$ is negative, the second-order work rate may or may not be negative, and thus the relations may be unstable. Consideration of $\dot{\sigma}_{ij} \dot{\varepsilon}_{ij}$ is necessary to establish whether instability is present.

**YIELD FUNCTIONS FOR MODEL**

The yield functions which make up the complete model are written in terms of the invariants $\sigma_m$ and $s$. The elastic domain in the $s$ and $\sigma_m$ space (note that $s \geq 0$) is bounded by three distinct yield surfaces, as shown in Fig. 1; these are the Drucker-Prager failure surface, the cap and the tension cutoff. Both the failure surface and the tension cutoff are represented as yield surfaces; this is clearly only a first approximation of the real behavior.

The Drucker-Prager yield condition (5) is defined by

$$F_1 = \alpha \sigma_m + s - k = 0.$$  \hspace{1cm} (16)

The constants $\alpha$ and $k$ are related to the angle of friction and the cohesion of the material, respectively. The function $F_1$ depends only on the stress invariants, and thus remains fixed in stress space.

In our particular model, we have chosen a parabolic cap defined by
FIG. 1.—Yield Surface of Plasticity Model

FIG. 2.—Nonlinear Hardening Rule for Cap Yield Surface

FIG. 3.—Stress Space Behavior of Compression and Tension Vertices
\[ F_2 = -(\sigma_m - \sigma_m^0) + R^2 s^2 = 0 \]  \hspace{1cm} (17)

The constant \( R \) is a shape factor; when \( R \) is set equal to zero, the plane cap used by Bathe, et al. (1) is recovered. The hardening parameter \( \sigma_m^0 \) depends on the plastic volume strain \( \varepsilon_p^0 \) which has occurred since the initial instant. Let \( \bar{\varepsilon}_p^0 \) denote the initial degree of compaction; the current degree of compaction is then

\[ \bar{\varepsilon}_p = \bar{\varepsilon}_p^0 + \varepsilon_p \]  \hspace{1cm} (18)

\[ \bar{\varepsilon}_p = W (1 - e^{D\sigma_m^0}) \]  \hspace{1cm} (19)

This relation is shown diagramatically in Fig. 2, where the significance of the constants \( W \) and \( D \) can be appreciated. The cap can translate along the \( \sigma_m \) axis, and can move either to the left or the right in Fig. 1.

The tension cutoff is regarded as part of the yield surface, given by

\[ F_3 = \sigma_m - T = 0 \]  \hspace{1cm} (20)

in which \( T \) is the maximum value that the mean hydrostatic tension \( \sigma_m \) can attain. This yield surface also remains fixed in the stress space.

**INVARIANT FORM OF CONSTITUTIVE RELATIONS**

The constitutive equations, in which stress rates are given in terms of strain rates, can be most simply appreciated if we work with the stress invariants, \( \sigma_m \) and \( s \), and the conjugate strain rate quantities, \( \varepsilon_p \) and \( \dot{\varepsilon} \), as defined in Eqs. 9-15. We shall limit attention to the derivation of the equations for the case when yielding occurs on both the Drucker-Prager surface and the cap. This state is defined by the conditions \( F_1 = 0; F_2 = 0; \) and \( F_3 < 0 \). For this given state we can identify four cases of loading and unloading, as shown in Fig. 3:

1. \( \dot{F} < 0, \dot{F}_2 < 0 \); elastic unloading.
2. \( \dot{F}_1 < 0, \dot{F}_2 = 0 \); yielding on the cap, unloading from the D-P line.
3. \( \dot{F}_1 = 0, \dot{F}_2 < 0 \); yielding on the D-P line, unloading from the cap.
4. \( \dot{F}_1 = 0, \dot{F}_2 = 0 \); loading on both yield surfaces. We shall see that the stress point may move along the D-P line in either direction.

We will consider case 4 in detail, in a manner essentially identical to that described by Maier (10). The yield function \( F_1 \) is given by Eq. 16, and, using Eqs. 18 and 19, it is convenient to rewrite \( F_2 \) as

\[ F_2 = -\sigma_m + R^2 s^2 + \frac{1}{D} \ln \left( a - \frac{\varepsilon_p^0}{W} \right) \]  \hspace{1cm} (21a)

in which \( a = 1 - \frac{\varepsilon_p^0}{W} \)  \hspace{1cm} (21b)

For loading on both surfaces, the plastic strain rates are given as (compare Eq. 14b):
\begin{align}
\varepsilon_p &= \lambda_1 \frac{\partial F_1}{\partial s} + \lambda_2 \frac{\partial F_2}{\partial s} = \lambda_1 + 2R^2 s \lambda_2 \quad \text{(22a)} \\
\varepsilon^p &= \lambda_1 \frac{\partial F_1}{\partial \sigma_m} + \lambda_2 \frac{\partial F_2}{\partial \sigma_m} = \alpha \lambda_1 - \lambda_2 \quad \text{(22b)}
\end{align}

with \(\lambda_1 \geq 0\) and \(\lambda_2 \geq 0\).

The total strain rates are

\begin{align}
\dot{\varepsilon} &= \frac{s}{G} + \lambda_1 + 2R^2 s \lambda_2; \quad \dot{\varepsilon}^p = \frac{\dot{\sigma}_m}{K} + \alpha \lambda_1 - \lambda_2 \quad \text{(23)}
\end{align}

and thus, inverting, we have

\begin{align}
\dot{s} &= G(\dot{\varepsilon} - \lambda_1 - 2R^2 s \lambda_2); \quad \dot{\sigma}_m = K(\dot{\varepsilon}^p - \alpha \lambda_1 + \lambda_2) \quad \text{(24)}
\end{align}

The condition for simultaneous loading on \(F_1\) and \(F_2\) is

\begin{align}
\dot{F}_1 &= \alpha \dot{\sigma}_m + \dot{s} = 0 \quad \text{(25a)} \\
\dot{F}_2 &= -\dot{\sigma}_m + 2R^2 s \dot{s} - \frac{1}{WD} \left( a - \frac{\varepsilon^p}{W} \right) \dot{\varepsilon}_p = 0 \quad \text{(25b)}
\end{align}

We now use Eq. 25a to express \(s\) in terms of \(\dot{\sigma}_m\), and, with Eq. 22b, solve Eq. 25b for \(\alpha \lambda_1 - \lambda_2\) in terms of \(\dot{\sigma}_m\):

\begin{align}
\alpha \lambda_1 - \lambda_2 &= -WD \left( a - \frac{\varepsilon^p}{W} \right) (1 + 2R^2 s \alpha) \dot{\sigma}_m \quad \text{(26)}
\end{align}

The second part of Eq. 24 then gives

\begin{align}
\dot{\sigma}_m &= \frac{KH}{H - K(1 + 2R^2 s \alpha)} \dot{\varepsilon}_p \quad \text{(27a)}
\end{align}

in which

\begin{align}
H &= \frac{1}{WD \left( a - \frac{\varepsilon^p}{W} \right)} \quad \text{(27b)}
\end{align}

Eq. 25a now gives

\begin{align}
\dot{s} = -\alpha \dot{\sigma}_m &= \frac{-\alpha KH}{H - K(1 + 2R^2 s \alpha)} \dot{\varepsilon}_p \quad \text{(27c)}
\end{align}

The requirement that \(\lambda_1 \geq 0\) and \(\lambda_2 \geq 0\) provides the conditions for loading. We now solve for \(\lambda_1\) and \(\lambda_2\) from Eq. 24, using Eq. 27; after some arithmetic, we find

\begin{align}
\lambda_1 &= \frac{1}{1 + 2R^2 s \alpha} \left( \frac{2R^2 s G K - \frac{\alpha KH}{1 + 2R^2 s \alpha}}{G K - H G + 2G K s R^2} \right) \dot{\varepsilon}_p; \\
\lambda_2 &= \frac{\alpha}{(1 + 2R^2 s \alpha)} \left( \frac{G K + \frac{\alpha^2 H K}{1 + 2R^2 s \alpha}}{G K - H G + 2G K s R^2} \right) \dot{\varepsilon}_p \quad \text{(28)}
\end{align}
The conditions for loading on both yield surfaces at the vertex are $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ in Eq. 28. The expressions can be simplified with extensive algebraic manipulation (which we shall not give in detail). Thus, the constitutive equations for $F_1 = 0$, $F_2 = 0$ and $F_3 < 0$, and $\bar{F}_1 = 0$ and $\bar{F}_2 = 0$ are given by

$$
\begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix} =
\begin{bmatrix}
0 & -\frac{\alpha KH}{H - K(1 + 2R^2 s\alpha)} \\
0 & \frac{KH}{H - K(1 + 2R^2 s\alpha)}
\end{bmatrix}
\begin{bmatrix}
\dot{\varepsilon} \\
\dot{\varepsilon}_v
\end{bmatrix}
$$

for $\dot{\varepsilon} + \frac{(4R^4 s^2 GK\alpha + 2R^2 sGK - \alpha KH)}{(GK - GH + 2R^2 s\alpha GK)} \dot{\varepsilon}_v \geq 0$ ....................................... (29a)

and $\dot{\varepsilon} - \frac{(GK + 2R^2 s\alpha GK + HK\alpha^2)}{\alpha(GK - GH + 2R^2 s\alpha GK)} \dot{\varepsilon}_v \geq 0$ ....................................... (29b)

Note that the stress invariant rates depend only on the volume strain rate, and not on the shear strain rate. Of course, the shear strain rate is not zero; it is given in Eq. 23, and it can be seen that the plastic shear strain rate will be non-negative for the condition $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. If the volume strain rate is zero, the stress rates are both zero, and plastic shear deformation may take place at constant stress. If the total volume strain rate is negative, the stress point moves along the Drucker-Prager line in Fig. 3 to the right, pushing the cap ahead of it. If, on the other hand, the total volume strain rate is positive, the stress point moves along the Drucker-Prager line to the left, pulling the cap behind it. In the latter case the relations are not necessarily stable, since

$$
\dot{\varepsilon} + \dot{\sigma}_m \dot{\varepsilon}_v = \frac{KH(\dot{\varepsilon}_v^2 - \alpha \dot{\varepsilon}_v \dot{\varepsilon})}{H - K(1 + 2R^2 s\alpha)}
$$

is not necessarily non-negative when $\dot{\varepsilon}_v > 0$ and $\dot{\varepsilon} \geq 0$.

Before commenting further on these relations, we shall complete the set of equations for the response at the vertex. First, we treat case 3 where yielding takes place on the Drucker-Prager yield surface only, i.e., $F_1 = 0$, $F_2 = 0$, and $F_3 < 0$, and $\bar{F}_1 = 0$ and $\bar{F}_2 < 0$. Nonzero plastic strain rates are possible, with

$$
\dot{\varepsilon}^p = \lambda_1 \frac{\partial F_1}{\partial s} = \lambda_1; \quad \dot{\varepsilon}_v^p = \lambda_1 \frac{\partial F_1}{\partial \sigma_m} = \alpha \lambda_1
$$

The condition for loading (i.e., $\lambda_1 \geq 0$) is

$$
\dot{F}_1 = \alpha \dot{\sigma}_m + \dot{s} = 0
$$

The total strain rates are

$$
\dot{\varepsilon} = \frac{\dot{s}}{G} + \lambda_1; \quad \dot{\varepsilon}_v = \frac{\dot{\sigma}_m}{K} + \alpha \lambda_1
$$

which give the stress rates as

$$
\dot{s} = G(\dot{\varepsilon} - \lambda_1); \quad \dot{\sigma}_m = K(\dot{\varepsilon}_v - \alpha \lambda_1)
$$
These equations are now substituted into Eq. 32 to give $\lambda_1$:

$$\lambda_1 = \frac{G}{G + \alpha^2 K} \dot{\varepsilon} + \frac{\alpha K}{G + \alpha^2 K} \dot{\varepsilon}_v \quad \ldots \quad (35)$$

Since it is required that $\lambda_1 \geq 0$, Eq. 35 also gives the condition for loading. Thus, on substituting Eq. 35 into Eq. 34 for $\lambda_1 \geq 0$, we get, for $F_1 = 0$, $F_2 = 0$ and $F_3 < 0$, and for $\ddot{F}_1 = 0$ and $\ddot{F}_2 < 0$:

$$\begin{bmatrix} \dot{s} \\ \dot{\sigma}_m \end{bmatrix} = \begin{bmatrix} G - \frac{G^2}{G + \alpha^2 K} & -\frac{\alpha G K}{G + \alpha^2 K} \\ \frac{\alpha G K}{G + \alpha^2 K} & K - \frac{K^2 \alpha^2}{G + \alpha^2 K} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon} \\ \dot{\varepsilon}_v \end{bmatrix} \quad \ldots \quad (36a)$$

for $Ge + \alpha K \dot{\varepsilon}_v \geq 0 \ldots \quad (36b)$

and

$$\dot{\varepsilon} - \frac{(GK + 2 R^2 s \alpha G K + HK \alpha^2)}{\alpha (GK - GH + 2 R^2 s \alpha G K)} \dot{\varepsilon}_v < 0, \quad (\ddot{F}_2 < 0) \quad \ldots \quad (36c)$$

Secondly, consider case 2, where yielding is associated with the cap, i.e., $F_1 = 0$, $F_2 = 0$ and $F_3 < 0$, and $\ddot{F}_1 < 0$, and $\ddot{F}_2 = 0$.

From Eq. 14b the plastic strain rates are

$$\varepsilon_p^\nu = \lambda_2 \frac{\partial F_2}{\partial s} = 2 R^2 s \lambda_2; \quad \varepsilon_p^\nu = \lambda_2 \frac{\partial F_2}{\partial \sigma_m} = -\lambda_2 \ldots \quad (37)$$

with $\lambda_2 \geq 0$. The total strain rates are then

$$\dot{\varepsilon} = \frac{\ddot{s}}{G} + 2 R^2 s \lambda_2; \quad \dot{\varepsilon}_v = \frac{\ddot{\sigma}_m}{K} - \lambda_2 \ldots \quad (38)$$

and thus, on inverting, we have

$$\dot{s} = G(\dot{\varepsilon} - 2 R^2 s \lambda_2); \quad \dot{\sigma}_m = K(\dot{\varepsilon}_v + \lambda_2) \ldots \quad (39)$$

The condition for loading is

$$\ddot{F}_2 = -\dot{\sigma}_m + 2 R^2 s \ddot{s} - H \dot{\varepsilon}_p^\nu = 0 \ldots \quad (40)$$

We now substitute Eqs. 37 and 39 into Eq. 40 in order to determine $\lambda_2$:

$$\lambda_2 = \frac{1}{K + 4 R^4 s^2 G - H} (2 R^2 s G \dot{\varepsilon} - K \dot{\varepsilon}_v) \ldots \quad (41)$$

The denominator in this expression is always positive, and thus the numerator gives the sign of $\lambda_2$. On substituting back into Eqs. 39, we thus find the cap constitutive equations for $F_1 = 0$, $F_2 = 0$ and $F_3 < 0$, and $\ddot{F}_1 < 0$ and $\ddot{F}_2 = 0$

$$\begin{bmatrix} \dot{s} \\ \dot{\sigma}_m \end{bmatrix} = \begin{bmatrix} G - \frac{4 R^4 s^2 G^2}{K + 4 R^4 s^2 G - H} & \frac{2 R^2 s G K}{K + 4 R^4 s^2 G - H} \\ \frac{2 R^2 s G K}{K + 4 R^4 s^2 G - H} & K - \frac{K^2}{K + 4 R^4 s^2 G - H} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon} \\ \dot{\varepsilon}_v \end{bmatrix} \quad \ldots \quad (42a)$$

for $2 R^2 s G \dot{\varepsilon} - K \dot{\varepsilon}_v \geq 0 \ldots \quad (42b)$
FIG. 4.—Total Strain Rate Space Behavior of Compression Vertex

Finally, for case 1, representing elastic unloading, we have the elastic constitutive relation defined by Eq. 14a. Thus, for $F_1 = 0$, $F_2 = 0$ and $F_3 < 0$, and $F_1 < 0$ and $F_2 < 0$:

$$\dot{\epsilon} + \frac{(4R^2s^2GK\alpha + 2R^2sGK - \alpha KH)}{(GK - Gh + 2R^2sGK)} \dot{\epsilon}_v < 0, \quad (F_1 < 0) \quad \ldots \quad (42c)$$

The conditions which separate the various loading and unloading cases are shown diagrammatically in Fig. 4, in terms of total strain rate. This diagram shows that, for $F_1 = 0$ and $F_2 = 0$, we have a complete set of relations. In cases 1, 2 and 3, the constitutive relations involve positive definite symmetric matrices, and in such cases, Drucker’s stability pos­tulate (4,6) is satisfied, since $G\dot{e} + \alpha K\dot{\epsilon}_v < 0$ for all $\dot{\epsilon}$ and $\dot{\epsilon}_v$ in these regimes. In all cases (including case 4), the plastic strain rate vector lies in the fan defined by adjacent normals at the singular point defining the intersection of $F_1 = 0$ and $F_2 = 0$ in stress space.

The regime of behavior described in case 4, Eq. 29, is of most interest. This provides the essential element that unlimited plastic shear deformation is possible. From a computational point of view, however, the equations have the disadvantage that they may permit an instability ($\dot{\epsilon} + \dot{\sigma}_m \dot{\epsilon}_v < 0$) when $\dot{\epsilon}_v > 0$. This leads to difficulties and instabilities in the solution, and forms of the constitutive equations in which the instability does not occur are of interest. We shall discuss two such cases.

In the model of Bathe, et al. (1), the values of $\lambda_1$ and $\lambda_2$, for the case when both yield surfaces are active, are taken as the sum of solutions for $F_1$ active alone and $F_2$ active alone; adding Eqs. 35 and 41 gives

$$\lambda_1 = \frac{G}{G + \alpha^2 K} \dot{\epsilon} + \frac{\alpha K}{G + \alpha^2 K} \dot{\epsilon}_v;$$
\[ \dot{\epsilon} = \frac{2R^2sG}{(K + 4R^4s^2G - H)} \dot{\epsilon} - \frac{K}{(K + 4R^4s^2G - H)} \dot{\epsilon}_v \]  

\[ \lambda_2 = \frac{2R^2sG}{(K + 4R^4s^2G - H)} \dot{\epsilon} - \frac{K}{(K + 4R^4s^2G - H)} \dot{\epsilon}_v \]  

The stress rate, strain rate equations then effectively come out to be made up of the terms in Eqs. 36 and 42:

\[
\begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix} = \begin{bmatrix}
G - \frac{G^2}{G + \alpha^2K} - \frac{4R^4s^2G^2}{K + 4R^4s^2G - H} & -\alpha G K + \frac{2R^2sG K}{G + \alpha^2K} \\
-\alpha G K + \frac{2R^2sG K}{G + \alpha^2K} & K - \frac{K^2}{G + \alpha^2K} - \frac{4R^4s^2G - H}{K + 4R^4s^2G - H}
\end{bmatrix} \begin{bmatrix}
\dot{\epsilon} \\
\dot{\epsilon}_v
\end{bmatrix}
\]  

It can be seen immediately from these equations that, for \( \dot{\epsilon} > 0 \) and \( \dot{\epsilon}_v = 0 \), the stress rates are not zero. A modified flow rule of the von Mises form is then introduced, permitting \( \dot{\epsilon}_v > 0 \) and \( \dot{\epsilon}_v = 0 \), and therefore \( \dot{s} = \dot{\sigma}_m = 0 \) for \( \dot{\epsilon} > 0 \) and \( \dot{\epsilon}_v = 0 \).

The additional assumption of a von Mises flow rule is not consistent with the form of the yield function, since it has been seen that no further assumptions are necessary. Eqs. 29a and 44b are substantially different in many respects, and could potentially lead to very different responses. One of the most important aspects is the suppression in Eq. 44b of any possible instability, since the matrix is positive definite.

The conditions which separate the various loading and unloading cases for the vertex behavior can be derived for the model of Bathe, et al. (1) in the same manner as that used for the present model. We have done this for the case of \( R = 0 \) (plane cap of Bathe, et al.) and the result is shown diagrammatically in Fig. 5. It can be seen immediately that the loading-unloading conditions for cases 1–3 are identical to the ones derived for the present model. However, the loading conditions for case 4 (yielding on both Drucker-Prager and cap yield surfaces) are given by \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \), where \( \lambda_1 \) and \( \lambda_2 \) are those given in Eq. 44a, i.e.

\[ G\dot{\epsilon} + \alpha K \dot{\epsilon}_v \geq 0 \]  

\[ \text{FIG. 5.—Bathe, et al. (1) Modification: Total Strain Rate Space Behavior of Compression Vertex} \]
FIG. 6.—Sandler and Rubin (15) Modification: Total Strain Rate Space Behavior of Compression Vertex

TABLE 1.—Constitutive Matrix

<table>
<thead>
<tr>
<th>State</th>
<th>$a_{11}$</th>
<th>$a_{12}$</th>
<th>$a_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Yielding on Drucker-Prager</td>
<td>$\frac{G}{G + \alpha^2 K}$</td>
<td>$\frac{\alpha G K}{G + \alpha^2 K}$</td>
<td>$\frac{\alpha G K}{G + \alpha^2 K}$</td>
</tr>
<tr>
<td>Yielding on cap</td>
<td>$\frac{4R^4 s^2 G^2}{K + 4R^4 s^2 G - H}$</td>
<td>$\frac{-2R^2 s G K}{K + 4R^4 s^2 G - H}$</td>
<td>$\frac{-2R^2 s G K}{K + 4R^4 s^2 G - H}$</td>
</tr>
<tr>
<td>Yielding on tension cutoff</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Yielding on Drucker-Prager and tension cutoff</td>
<td>G</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Yielding on Drucker-Prager model and cap</td>
<td>G</td>
<td>$\frac{\alpha K H}{H - K(1 + 2R^2 s^2 \alpha)}$</td>
<td>0</td>
</tr>
</tbody>
</table>
Comparing Eq. 36c to Eq. 45b, and Eq. 42c to Eq. 45a, it is clear that they do not match and that there are two gaps in the total strain rate space. This shows that the model of Bathe, et al. does not provide solutions for certain strain rate paths, due to the additional assumptions about the behavior of the plastic strain rate vector at the vertex, and due to the inconsistent calculation of the plastic multipliers, \( \lambda_1 \) and \( \lambda_2 \), in the case when both yield surfaces are active.

Let us now apply the vertex behavior suggested by Sandler and Rubin (15) to the framework of the present work. While Sandler and Rubin use an elliptical cap and a curved failure surface, the essential difference at the vertex is that Eq. 18 is modified by putting

\[
\dot{\varepsilon}_p^v = 0 \quad \text{if} \quad \dot{F}_1 = 0, \quad \dot{F}_2 = 0 \quad \text{and} \quad \dot{\varepsilon}_p^v > 0;
\]

otherwise \( \dot{\varepsilon}_p^v = \dot{\varepsilon}_v^p \) .................... (46)

This condition is introduced to prevent the cap from acting as a softening yield surface; it essentially prevents the cap-Drucker-Prager vertex from moving to the left in Fig. 3 with the stress point, and thus is in-

<table>
<thead>
<tr>
<th>( a_{22} ) (5)</th>
<th>Conditions (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( F_1 &lt; 0, F_2 &lt; 0, F_3 &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 &lt; 0, F_3 &lt; 0, G\varepsilon + \alpha K\varepsilon &lt; 0 ) \hspace{1cm} ( F_1 &lt; 0, F_2 &lt; 0, F_3 = 0, \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 &lt; 0, F_3 = 0, G\varepsilon + \alpha K\varepsilon &lt; 0, \dot{\varepsilon}_v &lt; 0 )</td>
</tr>
<tr>
<td>( \frac{K^2 \alpha^2}{G + \alpha^2 K} )</td>
<td>( F_1 &lt; 0, F_2 = 0, F_3 &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 &lt; 0, F_3 &lt; 0, 2R^2sG\varepsilon - K\varepsilon &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 &lt; 0, F_3 = 0, 2R^2sG\varepsilon - K\varepsilon &lt; 0, \dot{\varepsilon}_v &lt; 0 )</td>
</tr>
<tr>
<td>( \frac{K^2}{K + 4R^2s^2G - H} )</td>
<td>( F_1 = 0, F_2 &lt; 0, F_3 = 0, G\varepsilon + \alpha K\varepsilon &lt; 0, -\alpha \dot{\varepsilon} + \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 &lt; 0, F_3 = 0, G\varepsilon + \alpha K\varepsilon &lt; 0, \dot{\varepsilon}_v &lt; 0 )</td>
</tr>
<tr>
<td>( K )</td>
<td>( F_1 &lt; 0, F_2 &lt; 0, F_3 &lt; 0, \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 &lt; 0, F_3 = 0, \dot{\varepsilon}_v &lt; 0 )</td>
</tr>
<tr>
<td>( K )</td>
<td>( F_1 = 0, F_2 &lt; 0, F_3 = 0, \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 = 0, \dot{\varepsilon}_v &lt; 0 )</td>
</tr>
<tr>
<td>( \frac{KH}{H - K(1 + 2R^2s\alpha)} )</td>
<td>( F_1 = 0, F_2 = 0, \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 = 0, \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 &lt; 0, F_2 &lt; 0, F_3 = 0, \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 &lt; 0, F_2 &lt; 0, F_3 = 0, \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 = 0, F_3 = 0, \dot{\varepsilon}_v &lt; 0 ) \hspace{1cm} ( F_1 = 0, F_2 = 0, F_3 = 0, \dot{\varepsilon}_v &lt; 0 )</td>
</tr>
</tbody>
</table>

\[ \hom - ve_{p, \gamma} \geq 0 \quad (R = 0) \]
tended to suppress the possible instability apparent in Eq. 30 when $\dot{\epsilon}_v > 0$.

Since we are now dealing with a discontinuous evolution equation (Eq. 46), it is necessary to satisfy the constraints on $\dot{\epsilon}_p$ in addition to the normal loading-unloading constraints. The conditions that separate the various loading and unloading cases can be derived for this modified model, and are shown in Fig. 6. The conditions for cases 1–3 are identical to those obtained earlier, as would be expected. For case 4, Eq. 29b, $\lambda_1 \geq 0$ and the condition $\dot{\epsilon}_p \leq 0$ bound the range of total strain rates for which evolution Eqs. 18 and 46 give the same results. The volume plastic strain rate constraint can be expressed in terms of total volume strain using Eqs. 22b and 28. Thus

$$\dot{\epsilon}_p = \alpha \lambda_1 - \lambda_2 = \frac{(1 + 2 R^2 s \alpha) G K}{G K - H G + 2 G K \alpha s R^2} \dot{\epsilon}_v \leq 0 \quad \cdots \quad (47a)$$

which implies that $\dot{\epsilon}_v \leq 0 \quad \cdots \quad (47b)$

This means that, for $\lambda_1$ and $\lambda_2 \geq 0$ and zero or negative total volume strain rates, there is no difference between the Sandler-Rubin modification and Eq. 29; however, for $\lambda_1$ and $\lambda_2 \geq 0$ and positive total volume strain rates $\dot{\epsilon}_p > 0$, $\dot{\epsilon}_p = 0$, the equations will be different. In this case, using the conditions $F_1 = F_2 = 0$, and following the same procedures that were used previously, the constitutive equations are found to be

$$\dot{s} = \dot{\sigma}_m = 0 \quad \cdots \quad (48)$$

and the associated values of $\lambda_1$ and $\lambda_2$ are

$$\lambda_1 = \frac{1}{1 + 2 R^2 s \alpha} (\dot{\epsilon} + 2 R^2 s \dot{\epsilon}_v) \quad \cdots \quad (49a)$$

$$\lambda_2 = \frac{1}{1 + 2 R^2 s \alpha} (\alpha \dot{\epsilon} - \dot{\epsilon}_v) \quad \cdots \quad (49b)$$

Also, as an obvious consequence of Eq. 48, we have

$$\dot{\epsilon}_p = \dot{\epsilon}_v \quad \cdots \quad (50)$$

The total strain rate space bounds for this mode of behavior are given by $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ of Eq. 49, and $\dot{\epsilon}_v > 0$ (or $\dot{\epsilon}_p > 0$), which shows again a gap in the total strain rate space (Fig. 6). We see then that the modification suggested by Sandler and Rubin leads to an inconsistent model, the inconsistency arising from the choice of the evolution law for the history parameters.

The constitutive equations and loading and unloading constraints for the stress point at the intersection of the Drucker-Prager and tension cutoff yield surfaces (tension vertex) are treated in a manner similar to the compression vertex case, and will not be given here. The treatment of the constitutive equations relating to the cases when the stress point is on a single yield surface is straightforward, and is also omitted. However, a summary of the constitutive equations for the complete fully coupled model is given in Table 1, while a detailed description of the model can be found in Ref. 12.

In this summary, the constitutive equations for the cap model are writ-
ten in invariant form as
\[
\begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix} =
\begin{bmatrix}
G - a_{11} & -a_{12} \\
-a_{21} & K - a_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{\epsilon} \\
\dot{\epsilon}_v
\end{bmatrix}
\]  

(51)

in which the coefficients \( a_{11}, a_{12}, a_{21}, \) and \( a_{22} \) depend on the current state. The values of the coefficients are given in Table 1. In all cases except one, the coefficient matrix is symmetric, with \( a_{12} = a_{21} \), and semipositive definite, in the sense that \( \dot{s} \dot{\epsilon} + \dot{\sigma}_m \dot{\epsilon}_v \geq 0 \). The exceptional case was outlined in some detail in the foregoing.

**Constitutive Equations for Plane Problems**

The constitutive equations for the cap model have been given in invariant form, which permits a simple two-dimensional representation. For finite element implementation, however, a more general matrix form is required. For completeness, we present a generalization for the case of axisymmetric problems that can be reduced to plane stress and plane strain without difficulty.

The nonzero components of the total stress and strain are written in vector form as
\[
\sigma = (\sigma_{11} \sigma_{22} \sigma_{12} \sigma_{33})^T \]  

(52a)

\[
\epsilon = (\epsilon_{11} \epsilon_{22} \gamma_{12} \epsilon_{33})^T \]  

(52b)

in which \( \gamma_{12} = 2\epsilon_{12} \).  

(52c)

It is also convenient to identify the deviatoric components of stress and strain, in vector form, as
\[
\begin{bmatrix}
\sigma \\
\sigma_m
\end{bmatrix} = C \begin{bmatrix}
s \\
\sigma_m
\end{bmatrix}  
\]

(55a)

\[
\begin{bmatrix}
\dot{\epsilon} \\
\dot{\epsilon}_v
\end{bmatrix} = \dot{C} \begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix}  
\]

(55b)

We make use of the fact that
\[
\sigma_{11} + \sigma_{22} + \sigma_{33} = 0, \quad \text{and} \quad \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0 \]  

(54)

Simple transformations provide us with the total stress and strain components in terms of the deviatoric stress and strain vectors, the mean hydrostatical stress \( \sigma_m \) and the volume strain \( \epsilon_v \):

\[
C = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix} = C \begin{bmatrix}
\dot{s}_{11} \\
\dot{s}_{22} \\
\dot{s}_{12} \\
\dot{\sigma}_m
\end{bmatrix}  
\]

(55a)

\[
\begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix} = C \begin{bmatrix}
\dot{s}_{11} \\
\dot{s}_{22} \\
\dot{s}_{12} \\
\dot{\sigma}_m
\end{bmatrix} = C \begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix}  
\]

(55b)

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The deviatoric invariants can be written in terms of \( s \) and \( \mathbf{e} \). The invariant shear strain rate is

\[
\dot{e} = \frac{1}{s} s_{ij} \dot{e}_{ij} = \frac{1}{s} \mathbf{s}^T \mathbf{n} \dot{\mathbf{e}} \quad \ldots \quad (56a)
\]

in which \( \mathbf{n} = \begin{bmatrix} 2 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 2 \end{bmatrix} \) \ldots \quad (56b)

Similarly, it can be shown that

\[
s = \left( \frac{1}{2} s_{ij} s_{ij} \right)^{1/2} = \left( \frac{1}{2} \mathbf{s}^T \mathbf{n} \mathbf{s} \right)^{1/2} \quad \ldots \quad (57a)
\]

\[
\dot{s} = \frac{1}{2s} s_{ij} \dot{s}_{ij} = \frac{1}{2s} \mathbf{s}^T \mathbf{n} \dot{\mathbf{s}} \quad \ldots \quad (57b)
\]

The plastic shear strain rate components comprise contributions from yielding on \( F_1, F_2 \) and \( F_3 \) (not simultaneously, of course). We can thus put

\[
\dot{\mathbf{e}}^p = \lambda_1 \frac{\partial F_1}{\partial s} + \lambda_2 \frac{\partial F_2}{\partial s} + \lambda_3 \frac{\partial F_3}{\partial s} = \frac{\lambda_1}{2s} \frac{\partial F_1}{\partial s} \mathbf{s} + \frac{\lambda_2}{2s} \frac{\partial F_2}{\partial s} \mathbf{s} + \frac{\lambda_3}{2s} \frac{\partial F_3}{\partial s} \mathbf{s} \quad \ldots \quad (58)
\]

Substituting for \( F_1, F_2 \) and \( F_3 \) (Eqs. 16, 17 and 20), this becomes

\[
\dot{\mathbf{e}}^p = \left( \frac{\lambda_1}{2s} + R^2 \lambda_2 \right) \mathbf{s} \quad \ldots \quad (59a)
\]

whereas \( \dot{\mathbf{e}}^v = (\alpha \lambda_1 - \lambda_2 + \lambda_3) \) \ldots \quad (59b)

The stress rates are then

\[
\dot{s} = 2G(\mathbf{e} - \dot{\mathbf{e}}^p) = 2G \dot{\mathbf{e}} - 2G \left( \frac{\lambda_1}{2s} + R^2 \lambda_2 \right) \mathbf{s} ;
\]

\[
\dot{\sigma}_m = K(\mathbf{e}_v - \dot{\mathbf{e}}^v) = K \dot{\mathbf{e}}_v - K(\alpha \lambda_1 - \lambda_2 + \lambda_3) \ldots \quad (60)
\]

For any particular state, we substitute the appropriate expressions for \( \lambda_1, \lambda_2 \) and \( \lambda_3 \), and write Eq. 60 as

\[
\dot{s} = 2G \dot{\mathbf{e}} - \frac{a_{11}}{s} s \dot{\mathbf{e}} - \frac{a_{12}}{s} s \dot{\mathbf{e}}_v \quad \ldots \quad (61a)
\]

\[
\dot{\sigma}_m = K \dot{\mathbf{e}}_v - a_{21} \dot{\mathbf{e}} - a_{22} \dot{\mathbf{e}}_v \quad \ldots \quad (61b)
\]

Premultiplying Eq. 61a by \( (s^T \mathbf{n}/2s) \), Eqs. 61a and 61b become

\[
\dot{s} = (G - a_{11}) \dot{\mathbf{e}} - a_{12} \dot{\mathbf{e}}_v ; \quad \dot{\sigma}_m = -a_{21} \dot{\mathbf{e}} + (K - a_{22}) \dot{\mathbf{e}}_v \ldots \quad (62)
\]

which is identical to Eq. 51. Thus, the coefficients \( a_{11}, a_{12}, a_{21} \) and \( a_{22} \) can be taken directly from our examination of the invariant equations. This leaves us with the task of transforming Eqs. 61a and 61b into stress and strain rates \( \dot{\sigma} \) and \( \dot{\mathbf{e}} \).

Substituting from Eq. 56a, Eqs. 61a and 61b can be written in matrix
form as

\[
\begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix} = \begin{bmatrix}
2GI - \frac{a_{11}}{s^2} ss^T n & -\frac{a_{12}}{s} s \\
-\frac{a_{21}}{s} s^T n & K - a_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{e} \\
\dot{e}_v
\end{bmatrix} = D
\begin{bmatrix}
\dot{e} \\
\dot{e}_v
\end{bmatrix}
\] (63)

in which \( I = \text{a } 3 \times 3 \text{ unit matrix.} \)

We note that we may partition the matrices \( C \) and \( \hat{C} \) (Eq. 55) and write

\[
C = \begin{bmatrix}
I & h \\
-\hat{h}^T & 1
\end{bmatrix}; \quad \hat{C} = \begin{bmatrix}
n^{-1} & -\frac{1}{3} h \\
h^T & 1
\end{bmatrix}
\] (64a)

in which \( h^T = (1 \ 1 \ 0) \) .................. (64b)

It is also convenient to define

\[
\dot{s} = \frac{1}{s} s; \quad \dot{t} = \frac{1}{s} n s = n \dot{s} \] .................. (64c)

Using these relationships, we then form

\[
\sigma = C \begin{bmatrix}
\dot{s} \\
\dot{\sigma}_m
\end{bmatrix} = CD\hat{C} \dot{e} = D^* \dot{e}
\] (65a)

in which

\[
D^* = \begin{bmatrix}
I & h \\
-\hat{h}^T & 1
\end{bmatrix}\begin{bmatrix}
2GI - a_{11} \dot{s} \hat{t}^T & -a_{12} \dot{s} \\
-a_{21} \hat{t}^T & K - a_{22}
\end{bmatrix}\begin{bmatrix}
n^{-1} & -\frac{1}{3} h \\
h^T & 1
\end{bmatrix}
\] (65b)

In multiplying out Eq. 65b, we note several simple relationships:

\[
n^{-1} \hat{t} = \dot{s}; \quad \frac{1}{3} \hat{t}^T h = \dot{s} \hat{t} h; \quad h^T n^{-1} = \frac{1}{3} h^T \] ............ (66)

Carrying through the multiplications, and simplifying through the use of Eq. 66, we find that

\[
D^* = \begin{bmatrix}
2n^{-1} - a_{11} \dot{s} \dot{s}^T & -\frac{2}{3} G h + a_{11} \dot{s} \dot{s}^T h \\
-(a_{12} \dot{h}^T + a_{21} h \dot{s}^T) & -a_{12} \dot{s} + a_{21} h \dot{s}^T h \\
+(K - a_{22}) h h^T & +(K - a_{22}) h \\
-\frac{2}{3} G h^T + a_{11} h^T \dot{s} \dot{s}^T & \frac{4}{3} G + K - a_{11} (h^T \dot{s})^2 \\
+a_{12} h^T \dot{s} h^T - a_{21} \dot{s}^T & +(a_{12} + a_{21}) h^T \dot{s} - a_{22} \\
+(K - a_{22}) h^T & 
\end{bmatrix}
\] (67)

The matrix \( D^* \) is thus given explicitly. It reduces to the elasticity matrix
when there is no plastic deformation, i.e., when \( a_{11} = a_{12} = a_{21} = a_{22} = 0 \). Careful inspection will show that \( D^* \) is symmetric and semipositive definite when \( a_{12} = a_{21} \), but \( D^* \) is not symmetric and not necessarily positive definite when \( a_{12} \neq a_{21} \). The exceptional behavior for loading on the Drucker-Prager surface and the cap simultaneously is thus preserved in the full set of equations.

The expressions for the plastic multipliers, \( \lambda_1, \lambda_2 \) and \( \lambda_3 \), developed for the invariant case, are preserved in the generalized equations. These are necessary for the calculation of the plastic strain rates.

Summarizing, Eqs. 65a and 67, with Table 1, provide the complete constitutive equations for the cap model in the case of axisymmetric problems. The generalization of the invariant case is essentially straightforward, and the behavior noted in the invariant case is qualitatively preserved.

**NUMERICAL ILLUSTRATIONS**

The simplest way to illustrate the differences between the model developed in this paper and previously formulated cap models is to compare the output of numerical analysis in which the models are incorporated. The model described in this paper has been implemented in a finite element code for the solution of plane stress, plane strain and axisymmetric problems, and we have chosen to compare these results with the model described by Bathe, et al. (1), for which we have the software available. The major interest in the comparison is the behavior when both the Drucker-Prager yield surface and the cap are active, and the various possibilities that arise when the stress point moves along the Drucker-Prager yield surface in the direction of increasing hydrostatic tension. There are other differences between the writers' model and the Bathe, et al. model, but these are simply the result of different formulations and are of no consequence in the present context.

In order to provide insight into the major differences between the two models, the simple four-element plane strain model shown in Fig. 7(b) was subjected to a variety of loading paths. The stress and strain fields were uniform, and the block was first loaded so that the stress point was forced onto the Drucker-Prager cap vertex. [In Fig. 7(b), \( E = 100 \) ksi; \( v = 0.25 \); \( \alpha = 0.05 \); \( k = 0.1 \) ksi; \( W = -0.066 \); \( D = 0.78 \) ksi\(^{-1} \); and \( R = 0 \).] The way in which this was done is not important; significant are the subsequent loading paths, shown in Fig. 7(a). The strain increments are imposed, and fall within the boundaries defined by the various constraints shown in Fig. 4. The paths imposed are thus categorized as:

1. Path A—Loading on both yield surfaces with the cap moving out.
2. Path B—Loading on both yield surfaces with the cap stationary.
3. Path C—Loading on both yield surfaces with the cap moving in. This path involves potentially unstable behavior.
4. Path D—Loading on the Drucker-Prager yield surface, but unloading from the cap, with \( \varepsilon = 0 \). The cap does not keep up with the stress point.

The results of the various analyses, run both with the writers' model
FIG. 7.—Strain Paths Imposed on Model when Stress Point Is at Intersection of Cap and Drucker-Prager Model

and the Bathe, et al. model, are shown in Fig. 8 (plot of $s$ against $e$), Fig. 9 (plot of $\sigma_m$ against $\epsilon_v$) and Fig. 10 (plot $\sigma_m$ against $\epsilon^p_v$). For convenience, in examining these results, we shall refer to the absolute magnitude of $\sigma_m$ and $\epsilon_v$ when using the terms "increasing" and "decreasing."

For path A, the results obtained by the writers' model and the Bathe, et al. model are essentially the same, showing that for this path the differences in the models are slight. Both $s$ and $\sigma_m$ increase with $e$ and $\epsilon_v$.

For path B, the results are identical, as expected. The stresses remain constant during plastic shear flow with no plastic volume change.

For path C, the Bathe, et al. model assumes that only the Drucker-Prager yield surface is active, with $\alpha = \sigma$; however, the cap continues to move with the stress point, contradicting the imposed von Mises condition and illustrating the inconsistency introduced by the additional assumptions. As a result, the predictions of the two models are quite dif-
FIG. 8.—Deviator Stress versus Deviator Strain: (a) Writers' Model; (b) Bathe, et al. Model (1)

FIG. 9.—Hydrostatic Stress versus Volume Strain: (a) Writers' Model; (b) Bathe, et al. Model (1)
FIG. 8.—Deviator Stress versus Deviator Strain: (a) Writers' Model; (b) Bathe, et al. Model (1)

FIG. 9.—Hydrostatic Stress versus Volume Strain: (a) Writers' Model; (b) Bathe, et al. Model (1)
FIG. 10.—Hydrostatic Stress versus Plastic Volume Strain: (a) Writers’ Model; (b) Bathe, et al. Model (1)
FIG. 11.—Cap Movement and Its Consequences on Loading-Unloading-Reloading Cycle (Path E): (a) Writers' Model; (b) Bathe, et al. Model (1)

ferent. The slopes in the $s$-$e$ plot and the $\sigma_m$-$\epsilon_v$ plot are not the same, and there is substantial disagreement in the plot of $\sigma_m$ and $\epsilon_v$.

Path D is treated by the Bathe, et al. model in the same way as path C, whereas the writers' model predicts separation of the stress point and the cap, with plastic volume strain associated with plastic deformation on the Drucker-Prager yield surface. The differences are again most distinct in Fig. 10.

We see thus that differences occur in the two models when the stress path follows the Drucker-Prager yield surface with $s$ decreasing. This difference has further consequences, which can be seen in the loading, unloading, and reloading path shown in Fig. 11. In the second part of the path [shown as E2 in Fig. 7(a)], the writers' model predicts separation from the cap, and thus subsequent reloading is made up of linear elastic behavior followed by yielding on the cap and elastic plastic hardening behavior. The Bathe, et al. model, on the other hand, gives inconsistent behavior on unloading (i.e., no plastic volume strain accompanied by cap movement), and yielding on the cap occurs immediately on reloading. These differences are shown in Fig. 12, where $s$ and $\epsilon_v$ are plotted against $e$.

Note also that while path C falls into a potentially unstable regime, the particular path chosen is such that $s\dot{e} + \sigma_m \dot{\epsilon}_v$ is positive. Whether or not the path is unstable will depend on the previous loading history; for this particular case, unstable behavior occurs only for $0 \leq \dot{\epsilon}_v/\dot{e} \leq 0.33$, $877$
FIG. 12.—Path E Responses
whereas the domain of potentially unstable behavior is \( 0 \leq \dot{\epsilon}_o / \dot{\epsilon} \leq 0.88 \). This is shown diagrammatically in Fig. 13.

### Conclusions

The fully coupled Drucker-Prager cap model leads to a complete and fully consistent set of constitutive equations, as we have shown. These equations contain a regime within which the stability postulates are not necessarily satisfied, and the suppression of the unstable regime becomes a matter of interest.

However, it is shown that an arbitrary change in the coupling rules is not acceptable, since such a change can lead to an incomplete formulation. This suggests that the question of what coupling rules are permissible is open: Further work is necessary in order to ascertain whether the goal of a fully stable Drucker-Prager cap model is possible, and under just what coupling rules.

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### Appendix I.—Second-Order Work

The constitutive model in this paper is formulated in terms of the invariants, \( s \) and \( \sigma_m \), and \( \dot{\epsilon} \) and \( \dot{\epsilon}_o \). In terms of these invariants, the second-order work is \( \dot{s} \dot{\epsilon} + \dot{\sigma}_m \dot{\epsilon}_o \), and it is necessary to relate this expression to the actual second-order work

\[
\dot{\sigma}_{ij} \dot{\epsilon}_{ij} = \dot{s}_{ij} \dot{\epsilon}_{ij} + \frac{1}{3} \dot{\sigma}_{kk} \dot{\epsilon}_{kk} = \dot{s}_{ij} \dot{\epsilon}_{ij} + \dot{\sigma}_m \dot{\epsilon}_o \tag{68}
\]

The contribution from the deviators can be written in terms of elastic and plastic strain rate contributions:
\[ s_{ij} e_{ij} = s_{ij} e_{ij}^e + s_{ij} e_{ij}^\sigma \] \hspace{1cm} (69a)
\[ \dot{s}e = \dot{s}e^e + \dot{s}e^\sigma \] \hspace{1cm} (69b)

Using Eqs. 8b, 12 and 14b, we see that
\[ s_{ij} e_{ij} = \frac{\lambda}{2s} \frac{\partial P}{\partial s} s_{ij} = \frac{1}{2s} s_{ij} \dot{s}_{ij} + \frac{\lambda}{2s} \frac{\partial P}{\partial s} = \dot{s}e^p \] \hspace{1cm} (70)

Further, from Eqs. 2 and 14a:
\[ s_{ij} e_{ij} = \frac{1}{2G} s_{ij} \dot{s}_{ij} \] \hspace{1cm} (71a)

whereas
\[ \dot{s}e^e = \frac{1}{G} s^2 = \frac{1}{2G} \left( s_{ij} \dot{s}_{ij} \right) \left( s_{kl} \dot{s}_{kl} \right) \] \hspace{1cm} (71b)

It is evident from simple geometric arguments that
\[ \dot{s}e^e \leq s_{ij} e_{ij}^e \] \hspace{1cm} (72)
and thus
\[ \dot{s}e + \dot{s}_{m} e_{v} = \dot{s}_{ij} e_{ij} \] \hspace{1cm} (73)

**APPENDIX II.—REFERENCES**


