Ultimate strength of a composite cylinder subjected to three-point bending: correlation of beam theory with experiment

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Abstract

This paper investigates the ultimate bending response of a solid composite cylinder reinforced with uniaxially aligned continuous fibers. Experiments exhibited remarkable nonlinear load–deflections up to failure, indicating that a progressive failure process must have occurred in the cylinder. Thus, the failure of the outmost filament initially subjected to the maximum bending stress does not correspond to the ultimate failure, and additional loads should be still applicable to the cylinder. To reveal this progressive failure process, the cylinder is discretized into a number of parallel layers of different widths. Each layer is considered as a unidirectional lamina, whose overall load component is determined within the framework of classical beam theory. However, the lamina nonlinearity has been incorporated in the analysis using an instantaneous stiffness element defined by the micromechanics bridging model. The benefit of this model is in that only the constituent fiber and matrix properties are required in the analysis. As neither the first ply nor the last ply failure corresponds to the ultimate failure, in addition to the stress failure criterion used to detect the failure of an individual ply another deformation related parameter must be also employed to govern the ultimate failure and then to determine the ultimate strength. In the present case when a 16 layers discretization has been employed, the predicted fourth ply failure strength has been found to work for the ultimate strength and to correlate reasonably well with the experimental counterpart.

Keywords: Composite cylinder; Bending response; Beam theory; Ultimate strength

1. Introduction

The use of continuous fiber reinforced composite cylinders as primary-load carrying elements has been widely recognized in many engineering fields such as aerospace, automobile, chemical and energy, civil and infrastructure, sports and recreation, and even biomedical engineering. In biomedical applications, for example, an orthodontic archwire can be best developed using continuous fiber reinforced polymer matrix composite rod [1]. The fabrication of such an archwire is generally achieved by pultruding a continuous glass fiber yarn bundle impregnated with a biocompatible resin through a curing die [2,3]. As the composite archwire is primarily subjected to a flexural load, its bending behavior especially the ultimate bending strength must be well understood.

A few attempts have been made to obtain the effective properties of composite cylinders. Bhattacharyya and Appiah considered a single (softer) fiber cylinder embedded in an annular matrix cylinder subjected to lateral load and obtained its exact elastoplastic response solution [4]. Their results, however, are not directly applicable to the present case. The reason is that in a real application a large number of fiber filaments are gathered altogether and the failure of the outmost filament subjected to initially the maximum bending stress generally does not imply the ultimate failure of the composite cylinder. Starbuck performed a stress analysis for laminated composite cylinders under nonaxisymmetric loading and derived a closed-form solution for the cylinder responses only up to the first-ply failure [5]. There is limited work in the literature focusing on simulation of the ultimate bending behavior of a composite cylinder. Most has been on structural analyses within the framework of linearly elastic or geometrically nonlinear deformations [6,7]. For these latter analyses, the overall material parameters of the composite cylinders have to be pre-assumed.

The purpose of this paper is to determine the ultimate bending strength of a composite cylinder reinforced with
uniaxially aligned continuous fibers only based on the information of its constituent fiber and matrix properties. Experiments have shown that the composite cylinder exhibited remarkably nonlinear deformations on its load–deflection curve up to failure. This means that a progressive failure process must have occurred in the cylinder and that the ultimate filament initially subjected to the maximum bending stress does not correspond to the ultimate failure, since, otherwise, a linear load–deflection curve similarly to the situation under a longitudinal tension would have resulted. In order to track this progressive failure process, the cylinder is imaginatively separated into a number of parallel layers with different widths each of which is regarded as a uni-directional composite lamina. Thus, the analysis for the cylinder is converted to that for a laminated beam.

Classical beam theory assumes that only the axial normal stress on a cross-section dominates the bending response, and the stress is only correlated with the main curvature component. When applied to the laminate bending analysis, the beam theory has an advantage in simplicity. This theory is used for the analysis of the present cylinder. However, the nonlinear responses of the discretized lamina layers out of their elastic deformation range have been incorporated in the analysis. The instantaneous stiffness element of each lamina is determined by the micromechanics bridging model [8] only using the constituent fiber and matrix properties. As neither the first ply nor the last-ply failure corresponds to the ultimate failure [9,10], an additional deformation related parameter has to be employed to control the ultimate failure and then to determine the ultimate bending strength. In the present case when sixteen layers were used in the discretization the predicted fourth ply failure strength has been found to well represent the ultimate strength, and is in reasonable agreement with the measured data.

2. Analysis procedure

2.1. Discretization

A cross-sectional discretization for the cylinder is shown in Fig. 1, with a global coordinate system, \((x,y,z)\), where \(x\) is along the cylinder axis and \(x-z\) is the plane on which the bending load is applied. Suppose that the cross-section of the circular cylinder is discretized into \(N = 2n\) layers of an equal thickness, \(t\), given by (see Fig. 1)

\[
t = \frac{d}{N} = \frac{d}{(2n)},
\]

where \(d\) is the cylinder diameter. Referring to Fig. 1, the width of the \(i\)th layer, \(b_i\), is determined from

\[
b_i = \sqrt{d^2 - 4a_i^2} = \sqrt{d^2 - 4[(n - i)t + 0.5t]^2}, \quad i = 1, \ldots, n.
\]  

(2)

Now, the cylinder can be regarded as a laminated composite each layer of which may have a different width. Note that in the present case, the longitudinal directions of all the laminae coincide with the global \(x\) direction, whereas the other two transverse directions can be chosen along the \(y\) and \(z\) directions, respectively. Thus, the global coordinate system is coincident with the lamina local ones.

2.2. Beam theory

By neglecting the contribution of in-plane strains and the other two secondary curvatures in the classical laminate theory, the only normal stress increment on the beam cross-section due to the primary middle-plane bending curvature increment is given by

\[
d\sigma_{xx} = zQ_{xx}d\kappa_{xx}^0,
\]

where \(Q_{xx}\) denotes the reduced instantaneous stiffness element for plane stress-state. As the global coordinate system is coincident with all the local ones, \(Q_{xx}\) is simply the effective stiffness element of a lamina. The resultants of the bending stress given by Eq. (3) must be balanced with the overall applied load increments on the cross-section. Thus, we have

\[
dM_{xx} = \int_{-\frac{d}{2}}^{\frac{d}{2}} d\sigma_{xx} z b \, dz = C_{xx} d\kappa_{xx}^0, \quad \text{or} \quad d\kappa_{xx}^0 = \frac{dM_{xx}}{C_{xx}},
\]

(4)

where \(dM_{xx}\) is the bending moment increment on the cross-section and \(C_{xx}\) is the cylinder bending stiffness given by

\[
C_{xx} = \frac{1}{3} \sum_{k=1}^{N} Q'_{11}(z_k^1 - z_{k-1}^1)b_k.
\]

Here \(Q'_{11}\) is the instantaneous effective stiffness element of the \(k\)th lamina in its local system, and \(z_{k-1}\) and...
are the z-coordinates of the bottom and top surfaces of the lamina. As with the increase of the external load some layer fails before others, a stiffness discount must be applied to the failed lamina for the remaining analysis. Thus, the cylinder stiffness during the whole bending process is calculated through

$$C_{xx} = \frac{1}{3} \sum_{k=0}^{N} Q'_{11}(z_k - z_{k-1})b_k,$$

where $k_0$ represents the failed lamina.

Having obtained the curvature increment from Eq. (4), the cylinder deflection increment at the current load level is determined as

$$dw = \frac{dx_{xx}}{2} x(l-x),$$

where $l$ is the span of the beam between the two supports.

2.3. Lamina theory

2.3.1. Stiffness element

According to the bridging model, the lamina instantaneous compliance matrix is defined as [8]

$$[S_{ij}] = (V_f[S_{ij}^f] + V_m[S_{ij}^m][A_{ij}]) (V_f[1] + V_m[A_{ij}])^{-1},$$

where $[S_{ij}^f]$ and $[S_{ij}^m]$ are the instantaneous compliance matrices of the fiber and matrix materials, $V_f$ and $V_m$ are the fiber and matrix volume fractions, and $[A_{ij}]$ is a bridging matrix. Since only the longitudinal stress component has been assumed to exist for the lamina (Eq. (3)), there is no shear stress component in both the fiber and the matrix materials. Therefore, the instantaneous compliance matrix of the lamina has the same feature as its elastic counterpart and the stiffness element in Eq. (5), $Q_{11}$ is found to be

$$Q_{11} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}.$$

From Eq. (7), the explicit compliance elements are derived as (for detail, see Ref. [8])

$$S_{11} = (V_f S_{11}^f + V_m S_{11}^m) b_{11},$$

$$S_{12} = (V_f S_{12}^f + V_m S_{12}^m) b_{11} + [V_i S_{11}^f + V_m (S_{11}^m a_{12} + S_{12}^m a_{22})] b_{22},$$

$$S_{22} = (V_f S_{22}^f + V_m S_{22}^m) b_{12} + [V_i S_{22}^f + V_m (S_{22}^m a_{12} + S_{22}^m a_{22})] b_{22},$$

where

$$a_{11} = E_m/E_{11},$$

$$a_{22} = \frac{1}{2} \left(1 + \frac{E_m}{E_{11}}\right),$$

$$a_{12} = (S_{12}^f - S_{12}^m)(a_{11} - a_{22})/(S_{11}^f - S_{11}^m),$$

$$b_{11} = (V_f + V_m a_{11})^{-1},$$

$$b_{22} = (V_f + V_m a_{22})^{-1},$$

$$b_{12} = -V_m a_{12} b_{11} b_{22}.$$

In the above, $E_{11}^f$ and $E_{22}^m$ are the longitudinal and transverse moduli of the fiber material, and $E_m$ is the tangent to the stress–strain curve of the matrix material at the given load level. It must be realized that $E_m$ is generally different at tension from that at compression, depending on whether the current stress in the matrix is tensile or compressive. It is also noted that the matrix stress–strain curves should be obtained through bending tests.

2.3.2. Internal stresses and failure criterion

The averaged stress increment on the $k$th lamina reads (see Eq. (3))

$$d\sigma_{11} = \frac{z_k + z_{k-1}}{2} Q'_{11} dx_{xx},$$

which results in only longitudinal stress increments in the fiber and matrix materials, given by [8]

$$d\sigma^f_{11} = b_{11} d\sigma_{11} \quad \text{and} \quad d\sigma^m_{11} = a_{11} b_{11} d\sigma_{11}.$$

The total stresses are simply updated through

$$\sigma^f_{11} = \sigma^f_{11} + d\sigma^f_{11} \quad \text{and} \quad \sigma^m_{11} = \sigma^m_{11} + d\sigma^m_{11}.$$

No thermal residual stresses are taken into account in the present paper, and both $\sigma^f_{11}$ and $\sigma^m_{11}$ are zero initially. Now that the internal stresses in the constituents have been known, the lamina is considered to fail if any of the following conditions is fulfilled:

$$\sigma^f_{11} \geq \sigma^f, \quad \sigma^f_{11} \leq -\sigma^f_{ucc}, \quad \sigma^m_{11} \geq \sigma^m, \quad \text{and} \quad \sigma^m_{11} \leq -\sigma^m_{ucc}.$$

Here $\sigma^f$, $\sigma^f_{ucc}$ and $\sigma^m$, $\sigma^m_{ucc}$ are the ultimate tensile and compressive stresses of the fiber and the matrix materials, respectively, which should be obtained through bending tests. It is noted that $\sigma^f$ and $\sigma^f_{ucc}$ can be determined by applying loads longitudinally instead.

2.3.3. Constituent compliance elements

In the present paper, the fiber is considered as linearly elastic until rupture whose compliance matrix is defined by Hooke’s law and keeps unchanged during the whole loading process. On the other hand, the matrix material is elastic–plastic, whose compliance matrix is defined through the Prandtl–Reuss theory. Namely [11],

$$[S^c_{ij}] = \begin{cases} [S^e_{ij}], & \text{when } \sigma^m_{11} \leq \sigma^m_{Y}, \quad \text{or} \quad \sigma^m_{11} \geq -\sigma^m_{UC}, \\ [S^m_T], \text{ when } \sigma^m_{11} > \sigma^m_{Y}, \\ [S^m_C], \text{ when } \sigma^m_{11} < -\sigma^m_{Y}. \end{cases}$$

where $[S^e_{ij}]$ is the elastic component and $[S^m_{ij}]$ the plastic component. $\sigma^m_{Y}$ is the matrix yield strength at tension and $\sigma^m_{UC}$ is the matrix yield strength at compression. For
the present case where only one stress component exists, the plastic component is found to be [11]

\[
[S_{ij}]_p = \frac{E_m - E_{m}}{4E_mE_m} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Here, \( E_m \) is the matrix elastic modulus.

3. Modeling example

3.1. Experimental data

A UD composite cylinder has been fabricated using a bundle of five \( E \)-glass fiber yarns, each containing 200 fiber filaments (the filament diameter = 9 \( \mu \)m, Unitica Glass Fiber, Japan). The matrix used was a mixture of 68 wt.% of an epoxy resin, R50, and 32 wt.% of a hardener, H64, provided by Chemicrete Enterprise Pte Ltd (Singapore). A tube-shrinkage technique was applied for the cylinder fabrication, giving a diameter of 0.5 mm and a fiber volume fraction of 45%. More details regarding this fabrication have been reported elsewhere [12].

Three-point bending test was performed for the fabricated cylinder, with a span size of \( l = 14 \) mm following the load condition of an archwire [12]. A typical load–middle span deflection is plotted in Fig. 2, whereas averaged measured mechanical parameters are summarized in Table 1.

3.2. Constituent parameters

Both the \( E \)-glass fiber and the epoxy matrix used in this paper are considered as isotropic materials. From the material data sheet provided by the supplier, the fiber has some comparable mechanical parameters to those of the Silenka \( E \)-glass 1200tex fibers given in Ref. [13]. Thus, the elastic properties of the fiber were taken from Ref. [13] and are listed in Table 2. These parameters are considered to be the same until rupture at both tension and compression. The fiber tensile and compressive strengths, also given in Table 2, were retrieved from the uniaxial tensile and compressive strengths of the UD composite provided in Ref. [13], using formula summarized in Ref. [8].

Pure matrix panels of 6 mm thickness were prepared through resin-casting method and were cut to four-point bending test specimens using a water-cooled diamond saw. The measured tensile and compressive stress–strain curves under bending were approximated using four linear segments, from which the matrix hardening modulus at any load level was found to be

\[
E_m = \left( E_{m}^T \right)_i, \quad \text{when} \quad \left( \sigma_{Y}^m \right)_{i-1} \leq \sigma_{e}^m \leq \left( \sigma_{Y}^m \right)_i, \quad \text{with} \quad \left( \sigma_{Y}^m \right)_0 = 0,
\]

\[
E_m = \left( E_{m}^T \right)_4, \quad \text{when} \quad \sigma_{e}^m \geq \left( \sigma_{Y}^m \right)_4,
\]

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Measured properties of composite cylinder under 3-point bending test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Materials</td>
<td>Diameter ( d ) (mm)</td>
</tr>
<tr>
<td>GF/epoxy</td>
<td>0.5</td>
</tr>
<tr>
<td>*Standard deviation.</td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Mechanical properties of ( E )-glass fibers [13]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{11} ) (GPa)</td>
<td>( E_{22} ) (GPa)</td>
</tr>
<tr>
<td>74</td>
<td>74</td>
</tr>
</tbody>
</table>

Fig. 2. Comparison between measurement and prediction based on the beam theory.
where the tangential moduli and the critical stresses corresponding to tension and compression together with tensile and compressive strengths are summarized in Table 3. More details can refer to Ref. [12].

### 3.3. Layers of discretization

Predicted load–deflection curves by using different numbers of discretized layers are plotted in Fig. 3. In the predictions, the cylinder diameter was assumed to be 0.5 mm and a 45% fiber volume fraction was used. All of the predictions were essentially the same in the elastic part of the load–deflection curves. Discrepancies among them existed only for the later parts of the curves. It is seen that the prediction with 16-layers \((N = 16)\) of discretization was close to those with even more layers. Thus, the following results are all based on the 16-layers discretization for the cylinder cross-section.

### 3.4. Modeling result

Using the material parameters given in Tables 2 and 3 as input data, prediction was made for the load–deflection curve of the composite cylinder, with a diameter of 0.5 mm and a 45% fiber volume fraction, up to the sixth-ply failure. The predicted load–deflection is graphed in Fig. 2 for comparison with the experiment. The predicted bending curvatures at individual ply failures are also shown in the figure. Fig. 2 evidently indicates that the failure load corresponding to the fourth-ply failure should be regarded as the maximum load to be sustained by the present composite cylinder. This is because the predicted deflection at the fourth-ply failure is in the nearest close to but larger than the measured critical deflection, which was 1.89 mm. The predicted ultimate load together with the corresponding deflection is summarized in Table 4. It is seen from Fig. 2 and Table 4 that the beam theory used in this paper is satisfactory for the analysis of the cylinder ultimate bending response.

### 4. Discussions and conclusion

Due to anisotropic behavior, a laminated composite beam is much more difficult in analysis especially in an inelastic and ultimate failure analysis than an isotropic beam. However, when the beam is constructed using unidirectional laminae stacked in the same direction, this analysis can be much simplified. Closed-form formulae are available for the latter case, and are presented in this paper. To further realize the efficiency of this beam theory, a comparison between the predictions based on this theory and on the classical laminate theory [12] is shown in Fig. 4. The ultimate load together with the corresponding deflection predicted by the classical laminate theory is listed in Table 4. From Fig. 4, we can see that only some small difference existed in the

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**Table 3**

Mechanical properties of R50 epoxy matrix under bending

<table>
<thead>
<tr>
<th>(I)</th>
<th>Tensile properties (strength = 67.8 MPa)</th>
<th>Compressive properties (strength = 87.8 MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\sigma^T_m) (MPa)</td>
<td>(\sigma^C_m) (MPa)</td>
</tr>
<tr>
<td>1</td>
<td>28.8</td>
<td>35.8</td>
</tr>
<tr>
<td>2</td>
<td>48.9</td>
<td>52.9</td>
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<tr>
<td>3</td>
<td>63.4</td>
<td>68.4</td>
</tr>
<tr>
<td>4</td>
<td>67.8</td>
<td>87.8</td>
</tr>
</tbody>
</table>

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Fig. 3. Comparison between predictions with different discretized lamina layers.
load–deflection predictions between the two theories after the first-ply failure, both of which correlate well with the experiments. The predicted ultimate loads by both theories, indicated in Table 4, are in good agreement with the measured data although the classical laminate theory is slightly more accurate. In this regard, the beam theory is sufficiently applicable to the unidirectional composite beam analysis. On the other hand, significant discrepancy existed between the predicted curvatures based on the different theories, as shown in Fig. 4. To apply the beam theory, the curvature in the transverse ($y$) direction has been neglected. Furthermore, the contribution of the in-plane strains to the beam deflection has been ignored. Based on these assumptions, the calculated curvature in the longitudinal ($x$) direction was relatively large. However, based on the classical laminate theory both the predicted transverse curvature and the in-plane longitudinal strain (up to 7th-ply failure, not shown) were negative, resulting in a considerably smaller curvature in the longitudinal direction.

We have already seen that when a laminate is subjected to bending an additional controlling parameter must be provided in order to figure out its ultimate failure. It is important to obtain this parameter through a simple test on a beam mode and then apply it to more complicated laminate structural analyses. As the curvature calculated from the beam theory is significantly different from that based on the laminate theory, it seems that the curvature might not be suitable to serve as this additional controlling parameter. The beam deflection, 1.89 mm in the present case, can be a much better choice. Another choice would be a failed layer code. Based on the beam theory, the fourth ply failure corresponded to the ultimate failure. If we used the fourth ply failure as the additional controlling parameter and incorporated it into the classical laminate theory, the predicted ultimate load would have been still very close to the measured one, as indicated in Fig. 4.

### References


