Relations for single and product moments of record values from Gumbel distribution

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Abstract: In this paper some recurrence relations between the moments of record values from the Gumbel distribution are established. It is shown that using these recurrence relations, all the single and product moments of all record values can be obtained in a very simple recursive process.

1. Introduction

Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.) with cumulative distribution function (c.d.f.) \( F(x) \) and probability density function (p.d.f.) \( f(x) \). Set \( Y_n = \max(\min(X_1, \ldots, X_n)) \), \( n \geq 1 \). We say \( X_j \) is a lower (upper) record value of this sequence if \( Y_j < (>)Y_{j-1} \), \( j > 1 \). By definition \( X_j \) is a lower as well as an upper record value. One can transform from lower records to upper records by replacing the original sequence of r.v.’s by \( \{-X_j, j \geq 1\} \) or (if \( P(X_i > 0) = 1 \) for all \( i \)) by \( \{1/X_j, i \geq 1\} \); the upper record values of this sequence will correspond to the lower record values of the original sequence. We will confine our attention to just lower record values.

The indices at which the lower record values occur are given by record value times \( \{L(n), N \geq 1\} \) where \( L(n) = \min \{j \mid j > L(n-1), X_j < X_{L(n-1)}\} \), \( n \geq 1 \), with \( L(1) = 1 \). Chandler (1952) introduced record values and record value times. Feller (1966) gave some examples of record values with respect to gambling problems. Properties of record values of i.i.d. r.v.’s have been extensively studied in the literature. See Ahsanullah (1988), Arnold and Balakrishnan (1989), Arnold, Balakrishnan and Nagaraja (1992), Nagaraja (1988) and Nevzorov (1987) for recent reviews.

In this paper we will derive some recurrence relations for the single and product moments of record values from Gumbel distribution. This distribution is used in the analysis of greatest values of yearly floods, breaking strength of materials, aircraft loads, etc. (see Gumbel, 1958). For various properties of moments of order statistics from Gumbel and other related distributions, see Gumbel (1958), David (1981), and Balakrishnan and Cohen (1991). Similar recurrence relations for the single and product...
moments of record values from the exponential distribution have been established recently by Balakrishnan and Ahsanullah (1992).

2. Main results

The p.d.f. of \( X_{L(n)} \), \( n \geq 1 \), is given by

\[
f_n(x) = \frac{1}{(n-1)!} \left\{ \ln F(x) \right\}^{n-1} f(x), \quad -\infty < x < \infty.
\]

(2.1)

Let us consider the standard Gumbel population with p.d.f.

\[
f(x) = e^{-e^{-x}} e^{-x}, \quad -\infty < x < \infty,
\]

(2.2)

and c.d.f.

\[
F(x) = e^{-e^{-x}}, \quad -\infty < x < \infty.
\]

(2.3)

It is easy to see from (2.2) and (2.3) that for the Gumbel distribution

\[
f(x) = F(x) \{ \ln F(x) \}, \quad -\infty < x < \infty.
\]

(2.4)

We can make use of this property of the Gumbel distribution to derive some simple recurrence relations for the single and product moments of lower record values.

**Theorem 1.** For \( n \geq 1 \) and \( r = 0, 1, 2, \ldots \),

\[
E(X_{L(n)}^{r+1}) = E(X_{L(n)}^{r+1}) - \frac{(r+1)}{n} E(X_{L(n)})
\]

(2.5)

and, consequently, for \( n \geq 1 \) and \( r = 0, 1, 2, \ldots \) we can write

\[
E(X_{L(n)}^{r+1}) = E(X_{L(n)}) - (r+1) \sum_{p=1}^{n} E(X_{L(p)})/p.
\]

(2.6)

**Proof.** From (2.1), let us consider for \( n \geq 1 \) and \( r = 0, 1, 2, \ldots \),

\[
E(X_{L(n)}^{r}) = \frac{1}{(n-1)!} \int_{-\infty}^{\infty} x^r \left\{ \ln F(x) \right\}^{n-1} f(x) \, dx
\]

\[
= \frac{1}{(n-1)!} \int_{-\infty}^{\infty} x^r \{ \ln F(x) \}^n F(x) \, dx \quad \text{(by using (2.4)).}
\]

Upon integrating by parts treating \( x^r \) for integration and the rest of the integrand for differentiation, we simply obtain

\[
E(X_{L(n)}^{r}) = \frac{1}{(n-1)!(r+1)} \left[ n \int_{-\infty}^{\infty} x^{r+1} \{ \ln F(x) \}^{n-1} f(x) \, dx 
\right.
\]

\[
- \int_{-\infty}^{\infty} x^{r+1} \{ \ln F(x) \}^n f(x) \, dx \right]
\]

\[
= \frac{n}{r+1} \left[ \int_{-\infty}^{\infty} x^{r+1} \frac{1}{(n-1)!} \{ \ln F(x) \}^{n-1} f(x) \, dx
\right.
\]

\[
- \int_{-\infty}^{\infty} x^{r+1} \frac{1}{n!} \{ \ln F(x) \}^n f(x) \, dx \right]
\]

\[
= \frac{n}{r+1} \left( E(X_{L(n)}^{r+1}) - E(X_{L(n)}^{r+1}) \right).
\]
Upon rewriting the above equation, we derive the recurrence relation in (2.5). Then, by repeatedly applying the recurrence relation in (2.5), we simply derive the recurrence relation in (2.6).

**Remark 1.** The recurrence relation in (2.5) (or in (2.6)) can be used in a simple recursive way to compute all the single moments of all record values.

**Corollary 1.** By setting \( r = 0 \) in Theorem 1, we obtain in particular for \( n \geq 1 \) that

\[
E(X_{L(n+1)}) = E(X_{L(n)}) - \frac{1}{n},
\]

and

\[
E(X_{L(n+1)}) = E(X_{L(n)}) - \sum_{p=1}^{n} \frac{1}{p}. \tag{2.8}
\]

Next, the joint density function of \( X_{L(m)} \) and \( X_{L(n)} \), \( 1 \leq m < n \), is given by

\[
f_{m,n}(x, y) = \frac{1}{(m-1)!(n-m-1)!} \left( -\ln F(x) \right)^{m-1} \frac{f(x)}{F(x)} \cdot \left( -\ln F(y) + \ln F(x) \right)^{n-m-1} f(y), \quad -\infty < y < x < \infty. \tag{2.9}
\]

Once again, upon using the relation in (2.4), we can derive some simple recurrence relations for the product moments of record values along the lines of Theorem 1.

**Theorem 2.** For \( m \geq 1 \) and \( r, s = 0, 1, 2, \ldots \),

\[
E(X_{L(m)}^{r+1} X_{L(n)}^{s+1}) = E(X_{L(m)}^{r+1} X_{L(n)}^{s+1}) + \frac{r+1}{m} E(X_{L(m)}^{r} X_{L(n)}^{s+1}); \tag{2.10}
\]

for \( 1 \leq m < n - 2 \) and \( r, s = 0, 1, 2, \ldots \),

\[
E(X_{L(m)}^{r+1} X_{L(n)}^{s+1}) = E(X_{L(m+1)}^{r+1} X_{L(n)}^{s}) + \frac{r+1}{m} E(X_{L(m)}^{r} X_{L(n)}^{s}). \tag{2.11}
\]

**Remark 2.** The recurrence relations in (2.10) and (2.11) can be used in a simple recursive way to compute all the product moments of all record values.

**Corollary 2.** By repeated application of the recurrence relation in (2.11), with the help of the relation in (2.10), we obtain for \( n \geq m + 1 \) and \( r, s = 0, 1, 2, \ldots \) that

\[
E(X_{L(m)}^{r+1} X_{L(n)}^{s+1}) = E(X_{L(n)}^{r+1} X_{L(m)}^{s+1}) + \left( r + 1 \right) \sum_{p=m}^{n-1} \frac{1}{p} E(X_{L(p)}^{r} X_{L(n)}^{s}). \tag{2.12}
\]

By setting \( r = 0 \) and \( s = 1 \) in (2.12), we get for \( n \geq m + 1 \),

\[
E(X_{L(m)} X_{L(n)}) = E(X_{L(n)}^{2}) + E(X_{L(n)}) \cdot \sum_{p=m}^{n-1} \frac{1}{p}. \tag{2.13}
\]

Also, from Corollary 1 we see easily that

\[
E(X_{L(m)}) = E(X_{L(n)}) + \sum_{p=m}^{n-1} \frac{1}{p}. \tag{2.14}
\]
Corollary 3. For \( 1 \leq m \leq n - 1 \),

\[
\text{Cov}(X_{L(m)}, X_{L(n)}) = \text{Var}(X_{L(n)}). 
\]

**Proof.** From (2.13) and (2.14), we simply have

\[
\text{Cov}(X_{L(m)}, X_{L(n)}) = E(X_{L(m)}X_{L(n)}) - E(X_{L(m)})E(X_{L(n)}) 
= E(X_{L(n)}^2) + \sum_{p=m}^{n-1} \frac{1}{p} - E(X_{L(n)}) \left\{ E(X_{L(n)}) + \sum_{p=m}^{n-1} \frac{1}{p} \right\} 
= \text{Var}(X_{L(m)}). 
\]

\[
\square 
\]

Corollary 4. By repeated application of the recurrence relations in (2.10) and (2.11), we also obtain for \( m \geq 1 \) and \( r, s = 0, 1, 2, \ldots \),

\[
E(X_{L(m)}^{r+1}X_{L(m+1)}^{s+1}) = \sum_{p=0}^{r+1} \frac{(r + 1)^{(p)}}{m^p} E(X_{L(m+1)}^{r+p+1-s}), \quad (2.15) 
\]

and for \( 1 \leq m \leq n - 2 \) and \( r, s = 0, 1, 2, \ldots \),

\[
E(X_{L(m)}^{r+1}X_{L(n)}^{s+1}) = \sum_{p=0}^{r+1} \frac{(r + 1)^{(p)}}{m^p} E(X_{L(n)}^{r+p+1-s}), \quad (2.16) 
\]

where \((r + 1)^{(i)}\) is defined as

\[
(r + 1)^{(i)} = \begin{cases} 
1 & \text{when } i = 0, \\
(r + 1)r \cdots (r + 1 - i + 1) & \text{when } i \geq 1. 
\end{cases} \quad \square
\]

**Theorem 3.** For \( m \geq 1 \) and \( r, s = 0, 1, 2, \ldots \),

\[
E(X_{L(m)}^{r+1}X_{L(m+2)}^{s+1}) = E(X_{L(m)}^{r+1}X_{L(m+1)}^{s+1}) - (s + 1) E(X_{L(m)}^{r+1}X_{L(m+1)}^{s+1}) 
+ m \left\{ E(X_{L(m+1)}^{r+1}) - E(X_{L(m+1)}^{r+1}X_{L(m+2)}^{s+1}) \right\} \quad (2.17) 
\]

and for \( 1 \leq m \leq n - 2 \) and \( r, s = 0, 1, 2, \ldots \),

\[
E(X_{L(m)}^{r+1}X_{L(n)}^{s+1}) = E(X_{L(m)}^{r+1}X_{L(n)}^{s+1}) 
+ \frac{1}{n-m} \left\{ m \left\{ E(X_{L(m+1)}^{r+1}X_{L(n)}^{s+1}) - E(X_{L(m+1)}^{r+1}X_{L(n)}^{s+1}) \right\} 
- (s + 1) E(X_{L(m)}^{r+1}X_{L(n)}^{s+1}) \right\} \quad (2.18) \]

It is also of interest to mention here that this approach can be easily adopted to establish recurrence relations for product moments involving more than two record values. For example, by following the lines of Theorem 2, we can easily prove that for \( 1 \leq m_1 < m_2 < \cdots < m_k \) and \( r_1, r_2, \ldots, r_{k+1} = 0, 1, 2, \ldots \),

\[
F\left( \prod_{i=1}^{k-1} X_{L(m_i)}^{r_i} X_{L(m_i+1)}^{s_i+1} \right) = \left( \prod_{i=1}^{k-1} X_{L(m_i)}^{r_i} X_{L(m_i+1)}^{s_i+1} \right) + \frac{r_{k+1}}{m_k} E\left( \prod_{i=1}^{k} X_{L(m_i)}^{r_i} X_{L(m_i+1)}^{s_i+1} \right), \quad (2.19) 
\]
and that for $1 \leq m_1 < m_2 < \cdots < m_k \leq m_{k+1} - 2$ and $r_1, r_2, \ldots, r_{k+1} = 0, 1, 2, \ldots$,

$$E \left( \prod_{i=1}^{k-1} X_{L(m_i)}^{r_i+1} X_{L(m_k)}^{r_k+1} X_{L(m_{k+1})}^{r_{k+1}} \right) = E \left( \sum_{i=1}^{k-1} X_{L(m_i)}^{r_i} X_{L(m_k)}^{r_k+1} X_{L(m_{k+1})}^{r_{k+1}} \right) + \frac{r_k + 1}{m_k} E \left( \sum_{i=1}^{k+1} X_{L(m_i)}^{r_i} \right).$$

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References


Balakrishnan, N. and M. Ahsanullah (1992), Relations for single and product moments of record values from exponential distribution, submitted for publication.


