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GREEK AND VEDIC GEOMETRY

I. RITUAL GEOMETRY

It has been somewhat of an embarrassment to Euro-American classicists that the earliest Pre-Socratics, the Milesian or Ionian "natural philosophers" with which Greek philosophy began, were inhabitants of Asia Minor or what is now called Turkey. *The Encyclopedia of Philosophy* opens its account of the Milesian School with the party-line, ably put in words by W. K. C Guthrie: "Pre-Socratic philosophy differs from all other philosophy in that it had no predecessors." The account then takes a different turn: "They had predecessors of a sort, of course. It was not accidental that the first pre-Socratics were citizens of Miletus, a prosperous trading center of Ionian Greeks on the Asiatic coast, where Greek and Oriental cultures mingled" (Guthrie in Edwards 1967: VI, 441).

We hear no more about those "Oriental cultures" which is just as well because it would take us far afield: for not only does "Oriental" comprise more than Asia Minor, but the importance of Asia Minor to Greek civilization is not confined to philosophy. It includes, e.g., mathematics. The great Greek geometer Eudoxos worked there and Apollonius of the *Conics* came from the same area.

The dogma that all philosophy started around 600 B.C. in ancient Greece has never been accepted by a majority and what remains of its appeal is dwindling. Yet many moderns paradoxically combine that non-acceptance with the acceptance of the idea of *le miracle grec* according to which science originated in Greece and is, therefore, in essence, "Western." They manage to perform that balancing act by making the assumption that, e.g., most Indian philosophies are unrelated to science, unlike many of the Greek and – by facile extension – the later Euro-American varieties. I shall not discuss that assumption itself but even if it were correct, it cannot be due to any alleged absence of science in India: for the most telling argument against *le miracle grec* is, in fact, provided by the Indian geometry of the Vedic period as preserved in the *Śulbasūtras* of the last millenium B.C.

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Following a series of studies published by A. Seidenberg in the *Archive for History of the Exact Sciences* and elsewhere, most historians of science are now agreed that the geometries of Greece and Vedic India were basically similar and similarly related to ritual. Van der Waerden provides a summary:

> A. Seidenberg has pointed out that in Greek texts as well as in the *Śulbasūtras*, geometrical constructions were regarded important for ritual purposes, namely for constructing altars of given form and magnitude. In Greece this led to the famous problem of "doubling the cube," whereas in India it was not the volume but the area of the altar that was considered important. In both cases, one essential step in the altar constructions was the solution of the problem: *to construct a square equal in area to a given rectangle*. To solve this problem, exactly the same construction was used in Greece and in India, a solution based on the Theorem of Pythagoras. Also, the ideas about the religious importance of exact geometrical altar constructions were very similar in both countries (van der Waerden 1983: 11).

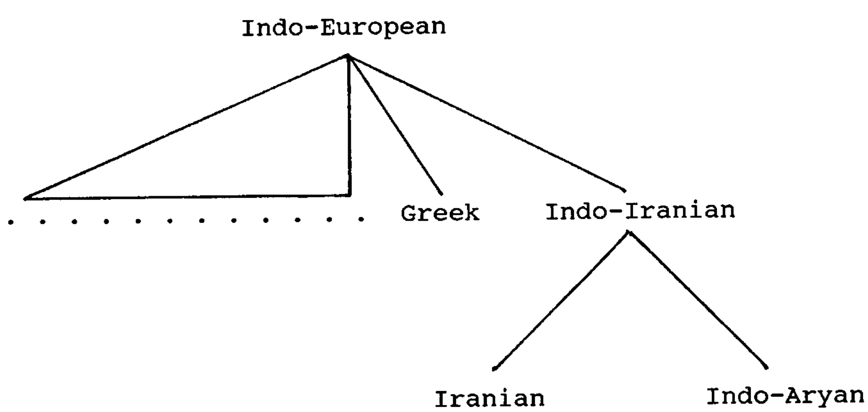
Some of these facts have long been known and can no longer be ignored even by Euro- (or Indo-)centric historians of science. In such cases, what J. D. Bernal called "arrogant ignorance" shifts ground. Unable to deny that the Theorem of Pythagoras occurs worldwide, some historians continue to believe (following, e.g., Sir Thomas Heath, the translator of Euclid's *Elements*: 1956, I: 363) that only the Greeks were capable of abstraction or providing *proofs*. This can be disproved with the help of Indian mathematics as Seidenberg (e.g., 1978, 1983) and van der Waerden (1983: 26 sq., adding China on p. 36) have shown (for more detailed discussion see Hayashi 1994: 122–123 and 1995: 72–77).

Van der Waerden has argued on heuristic grounds against "independent invention" as an explanation for such similarities. He prefers "diffusionism" as a working hypothesis, i.e., the idea that mathematical ideas arise only once and spread from a center. Much depends, of course, on how specific the mathematical ideas we study are. There may be no need to argue for the diffusion of elementary arithmetic truths such as $2 \times 2 = 4$. But in the case of the geometrical constructions of ancient Greece and India, the degree of specificity is high and independent invention therefore unlikely. Direct influences from Greece to India or vice versa during that period are equally improbable. Can we demonstrate a common origin?

It would seem natural to look for such an origin in Mesopotamia, situated between Greece and India, and especially among the Babylonians who possessed a highly developed mathematics at a much earlier period. Van der Waerden espoused this view in his well-known book *Science Awakening* of 1954. But Babylonian mathematics is different from the Indo-Greek variety: it was not *constructive*; it stressed *algebraic* and *computational* methods including fractions expressed through the sexagesimal number system. Seidenberg concluded that the common origin of Greek and Vedic mathematics must be pre-Babylonian and suggested Sumerian as a possible source. Actually, Sumerian mathematics represents an early phase of Mesopotamian mathematics, less developed than the Babylonian and different in several respects (Friberg 1990; Høyrup 1994: 21–22). Sumerian numerals "are *not* clearly sexagesimal. Neither are they constructed in a uniform way" (Friberg, p. 539). It is likely that Indian astronomy was influenced by Mesopotamia (Pingree 1973 and 1978), and that that Mesopotamian heritage included Sumerian (Pingree 1989). But the Vedo-Greek variety of geometry is different from all Mesopotamian mathematics: it was not inspired by celestial bodies but by altars.

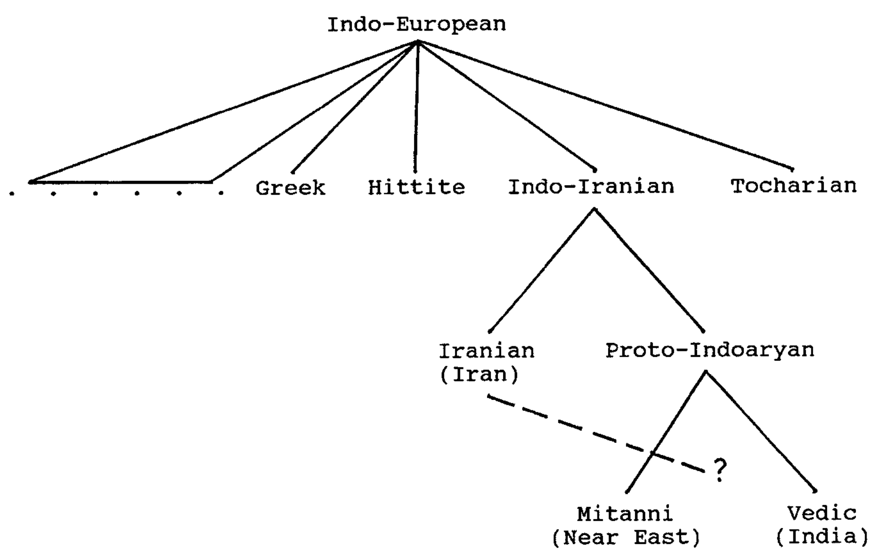
In his 1983 book, van der Waerden introduced another hypothesis which, from a linguistic point of view, is unsurprising and which had also occurred to Seidenberg (1983: 124–125) although he did not pursue it: Greek and Vedic geometry, like the Greek and Sanskrit languages, have a common Indo-European origin. Barring a brief mention of "the Danube region" (1983: 14), Van der Waerden did not address the next question: where in actual geography does that origin lie?

To answer that question is not simple and we must begin with a linguistic excursus on the place of Greek and Sanskrit within the family of Indo-European languages. To begin with Sanskrit, it belongs with Old Iranian to a subgroup of Indo-European called *Indo-Iranian*. Indo-Iranian has two branches: the Western (Iranian) and the Eastern (Indian or Indo-Aryan):



It has long been customary to divide Indo-European itself along similar lines into two groups, the Western and the Eastern, referred to respectively as *centum* and *satam* languages after the names of the numerals for "hundred" in Latin and Sanskrit. But there has always been a difficulty: Greek is in many respects more similar to Indo-Iranian *satam* languages than to the Western *centum* languages. Even the same innovations occurred (as in insects and birds developing wings): Greek and Indo-Iranian developed the augment in past tenses (Greek ἔφερε *ephere* "he carried," Sanskrit *ábharat*, from the Indo-European root *bhṛ* which survives in English "bear"). The augment is also found in Armenian but in no other Indo-European language (Burrow 1955: 15).

The *satam*/*centum* bifurcation had to be changed in scope or abandoned because of the discoveries of extinct Indo-European languages such as Tocharian (spoken in Central Asia), Hittite (spoken in Asia Minor or Turkey), and vestiges of Indo-Aryan or Proto-Indoaryan (Burrow 1973) found in the Near East such as "Mitanni Aryan" (spoken in Syria and Northern Mesopotamia). Some Mitanni names of deities are practically the same as the Vedic (e.g., Indra and Varuṇa) although they are not found elsewhere. The evidence for Indo-European from the Near East is not merely closer to Indo-Iranian than to any other Indo-European languages, but, *within* Indo-Iranian, closer to Indo-Aryan than to Iranian. In other words, there exists, within Indo-Aryan, a large gap between "East" and "West" with Iranian intervening as a wedge. Geography and linguistics seem to be inconsistent here for if we walk from the Near East to India – or vice-versa – we have to pass through Iran:



How can these facts be explained?

In order to keep this simplified account as simple as possible, I have not added dates. With the exception of Tocharian, for which the surviving evidence is late, most of the languages mentioned belong to the second millenium B.C. If we take account of this fact, everything can be explained by the hypothesis that the original "homeland" of the Indian and ancient Near Eastern Indo-European languages must be sought in the steppes along the Oxus river, now called the Amu Darya, which separates Turkmenistan and Uzbekistan, the area east of the Caspian Sea or Bactria and Margiana as they were called in classical times. From that region, several waves of people entered Iran: an early wave introduced a language that was or became Proto-Indoaryan; a later wave introduced Iranian which acted as a wedge, driving a few Indo-Aryan speakers to the Near East and the majority to India. The expulsion of Indo-Aryan by Iranian was boosted further by the reform of Zoroaster, who condemned the older Indo-Iranian religion. Some such account of an Iranian wedge has been accepted by most contemporary scholars from Burrow (1973) to Mallory (1989), Parpola (1993) and Diakonoff (who distinguishes three branches of Indo-Iranian, 1995: 474–475).

Where does that leave Greek? Greek resembles, but does not belong to Near-Eastern Indo-Aryan. As we have seen, it was not only spoken in Greece but also in Asia Minor. Asia Minor is adjacent to northern Mesopotamia where, some centuries earlier, an Indo-Aryan language appeared along with Indo-Aryan deities. If we make the simple assumption that the speakers of that language *brought along their rituals and ritual geometry*, that would explain why the same type of geometry came to flourish in India and Greece.

I shall strengthen this historical hypothesis later with the help of archeology, but the geography is clear and persuasive from the map of central Eurasia (Figure 1) which shows that Bactria/Margiana is about equally close to – or distant from – Asia Minor, Mitanni Syria and South Asia.

How do people "bring along their rituals and ritual geometry"? To explain this will take some time and must begin with a closer look at the ritual geometry of the Vedo-Greek variety itself. I shall first briefly pursue the ritual background and then discuss some of the geometry that is based upon it. Much of that latter information is also accessible in Seidenberg's articles in the *Archive for History of Exact Sciences*, van der Waerden 1983 or Joseph 1990.

Seidenberg (1983: 101–102) provides and discusses most of the information that is available on the Greek problem of "doubling the

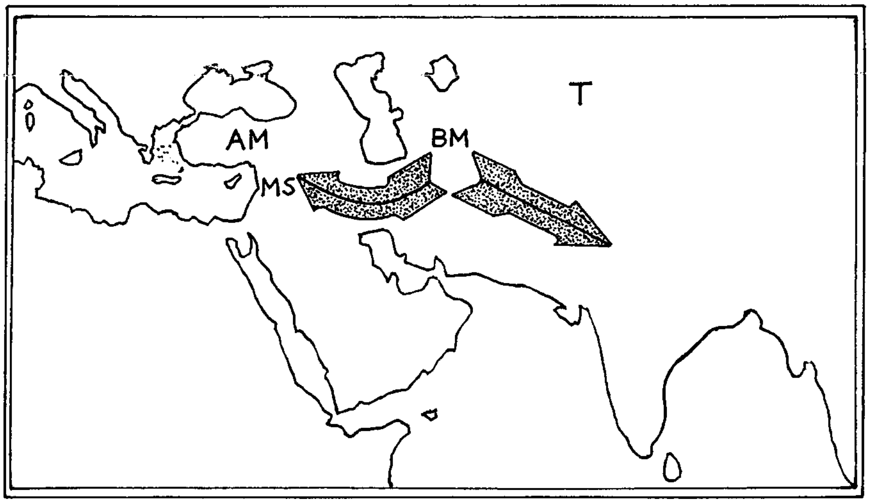


Figure 1. Eurasia. AM: Asia Minor; MS: Mitanni Syria; T: Tocharian; BM: Bactria-Margiana.

cube." Several legends dating from between the fifth and third century B.C. refer to the duplication of the size of an altar or royal tomb of that shape. The ritualists and architects realized that doubling the sides would not do the trick: for when the sides are doubled, the area is enlarged fourfold and the volume eightfold. Experts on geometry had to be consulted. But why should an altar be doubled in the first place?

The oracle at Delos prescribed this duplication in order to fight a plague. Seidenberg does not explain why this occurred to the oracle but refers to a passage in which Plato expresses his scorn for priests who use ritual for such purposes as expiating sins or harming an enemy:

> Mendicant prophets go to rich men's doors and persuade them that they have a power committed to them by the gods of making an attonement for a man's own or his ancestor's sins by sacrifices or charms, with rejoicings and feasts; and they promise to harm an enemy, whether just or unjust, at small cost; with magic arts and incantations binding heaven, as they say, to execute their will (*Republic* 364 b–c; transl. B. Jowett).

In the case of Vedic mathematics, almost all interesting results arose in connection with the constructions of one type of altar: the altar of the Agnicayana or "piling up of Agni," a bird-shaped construction consisting of *five* layers of *bricks*, each consisting of 200 bricks of different shapes arranged in such a way that the interstices between bricks in contiguous layers do not coincide except at the center. The *Śulbasūtras* mention different traditions of this construction and various shapes of altars, including the circular (also mentioned by *Baudhāyana*

Śrauta Sutra 17: 29, translated by Ikari and Arnold 1983 II: 668–671; cf. Pingree 1981: 3 sq. *ubi alia*). Several details are missing in these texts and were transmitted orally. They are preserved by the Nambudiri Brahmans of Kerala who have maintained, up to the present, three traditions of bird altars: the "Six-Tipped," "the Five-Tipped" and "the Square." For there is an important difference between ancient Greece and India: ancient Greek civilization is no longer a living tradition and there is nothing left of Greek ritual. In India, on the other hand, some of the ancient ritual and geometric knowledge survives (albeit in inaccessible corners) through unbroken chains of transmission as is demonstrated by precisely these oral traditions. The following brief description is based upon the Nambudiri tradition (cf. Staal 1982 and for a complete account: Staal et al. 1983).

In the first Nambudiri tradition, the Bird Altar with Six-Tipped Wings, the first layer looks as in Figure 2:

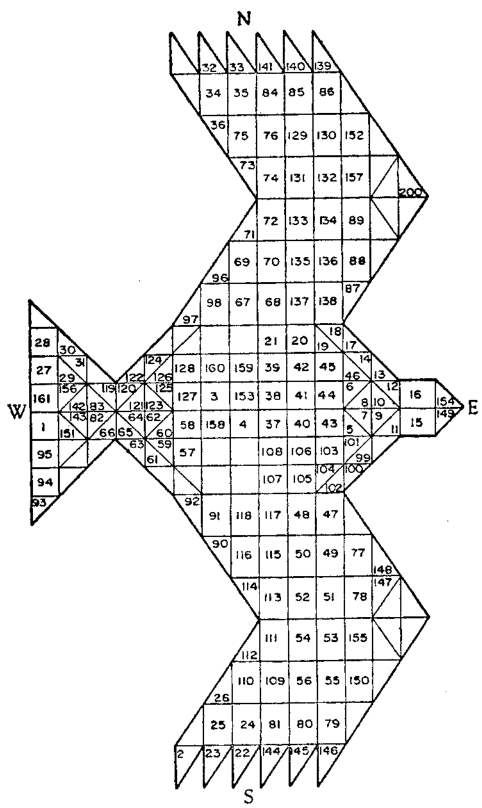


Figure 2. First layer of the Six-Tipped Altar.

In this figure, numbers indicate the (unexplained) order in which the bricks have to be consecrated with mantras (non-numbered bricks may be consecrated in any order).

In this first layer, which has the same configuration as the third and fifth, there are 38 squares, 58 rectangles (of two sizes) and 104 triangles (of two sizes). The second layer, not depicted here, has the same configuration as the fourth with 11 squares, 88 rectangles (of two sizes) and 201 triangles (of six sizes and five shapes). The sizes are constructed from a unit square with side one fifth the size of the Yajamāna or Ritual Patron. Some simple geometrical constructions are needed:

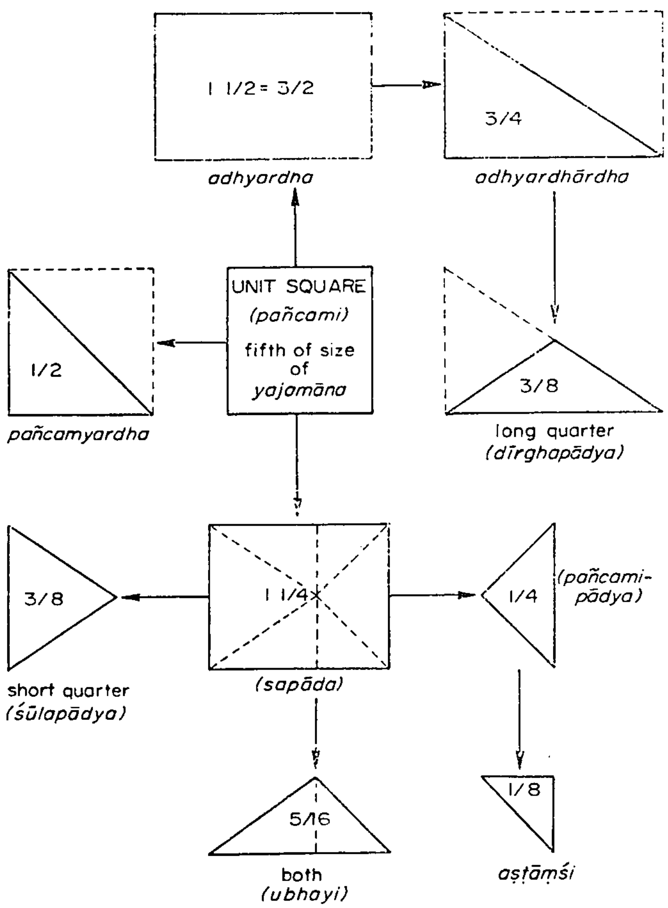


Figure 3. Sizes and shapes of bricks of the Six-Tipped Altar.

In the first, third and fifth layers of the Five-Tipped Bird Altar, there are 61 squares, 136 triangles (of two sizes) and three pentagons (of two sizes and three shapes). The second and fourth layers consist of 72 squares and 128 triangles (of two sizes). The constructions of these from the basic square is accordingly different. In the Square Bird Altar, which seems trivial at first, the construction is different again and not any simpler.

The mathematics of bricks is further constrained by other ritual requirements. The total area of each layer of the altar must be seven-and-a-half times a square *puruṣa*, i.e., a square of which the side is the size of the Yajamāna. Since the Yajamāna was measured in five units to arrive at the unit square of Figure 3, the square puruṣa is 25 times that unit square. The numbers, shapes and areas of the bricks are part of the oral tradition but the computation of the areas is not stated anywhere in clear terms. This is not surprising since the matter is far from obvious: for we should not forget that the Vedic Indians, like their Greek cousins, lacked simple expressions for numbers, whether integers or fractions. All calculation was done geometrically which is more complicated, especially for us, than if it were done in our modern notation. Only by adopting the latter can the number and area of bricks be easily expressed. For the first and third layers of the Six-Tipped Altar, it may be done as follows:

	NUMBER	AREA PER BRICK	AREA
square	38	1	38
1 1/4	2	1.25	2.5
1 1/2	56	1.5	84
1/2	60	0.5	30
3/4	44	0.75	33
TOTAL	200		187.5 = 7 1/2 × 25

The fifth layer is similar but there are 205 bricks, 10 of half thickness. The computation for the second and fourth layers are more complex but basically similar.

None of the evidence from the Agnicayana we have so far considered is in an obvious manner related to Greek geometry. Arrogant ignoramuses will be quick to point out that this "Agni" does not look like the deductive system of Euclid, which is true. The only Indian counterpart to Euclid is the derivational system of Pāṇini's Sanskrit grammar. It is in some respects more sophisticated than Euclid because it includes metarules and other special devices: Staal 1963, 1965, 1988: 143–160.

It is constructions such as those of the Agnicayana, however, that led to results of Vedic geometry that closely correspond to Greek geometry. For these results we have to look beyond the Agnicayana itself at the more advanced constructions discussed in the Śulbasūtras. For although these constructions are sometimes directly related to altar constructions, they are often independent and purely theoretical. The evolution of scientific geometry from ritual geometry is to some extent similar to that of Paninian grammar from earlier forms of Indian linguistics. Let us take a closer look at both.

II. FROM RITUAL TO SCIENCE

I shall begin with Indian linguistics where the earliest works are the Prātiśākhyas. They are early because of their structure and function, not because of the form in which they survive and which has already been influenced by Pāṇini's grammar of the fourth century B.C. (cf. Cardona 1976: 273–275). There is, as the name "Prātiśākhya" indicates, one of these treatises for each of the branches (*śākhā*) of the Veda. They are practical manuals that provide the rules for converting the *padapāṭha* (word-for-word recitation) of their own branch into its *saṃhitā* (continuous recitation). Some of these rules happened to be general rules that pertain to the language of all the Vedas; though variously formulated, they recur in most or all of the Prātiśākhyas. They also occur in Pāṇini's grammar which deals primarily with the language spoken during his time and in his part of country, a language that happened to be more or less identical with the language of late Vedic prose (i.e., the Brāhmaṇas and Sūtras) and came to be called Classical Sanskrit.

Pāṇini included cases where (earlier) Vedic deviated from Classical Sanskrit, but did not mention it when the same grammatical forms are exhibited by both languages. Hence Patañjali's remark on Pāṇini's grammar: *sarvavedapāriṣadam idaṃ śāstram*, "this science pertains to all the Vedas" – the motto of Paul Thieme's 1935 book "Pāṇini and the Veda" in which these distinctions were for the first time elucidated clearly in a modern language.

The *Śulba* Sūtras are attached to the *Śrauta* Sūtras, ritual sūtras that are practical manuals like the Prātiśākhyas: they provide the officiating priests with rules for the execution of their rites. Because of the proliferation of ritual, the subdivision into sūtras is more precise than that into śākhās: there are several sūtras within each śākhā. All Śulbasūtras are attached to Śrautasūtras of the Yajurveda, the ritual Veda *par excellence* and the core of the Vedic tradition.[1] It possesses two Prātiśākhyas (one for the Kṛṣṇa- and one for the Śukla-Yajurveda) and ten Śrautasūtras to six of which the six known Śulbasūtras are attached.

The Śrautasūtras generally exhibit rites that are specific to their own school. Others are shared by several manuals, e.g., many of the metarules (*paribhāṣā*) from which the science of ritual developed (Staal 1982). The Śulbasūtras share even more of their contents with each other: they deal with the geometry of altar construction which is the same even if the altar shapes are different. The relation of the Śulba- to the Śrautasūtras is, therefore, similar to that of Pāṇini to the Prātiśākhyas. They pertain to all the Vedas also. Their compilers did not continue to restrict themselves to their own ritual sūtra, but began to develop a more universal discipline. These manuals are therefore, as Thieme wrote of Pāṇini, not merely practical but scientific. One could go further and apply what David Hilbert wrote about the importance of scientific work in general: it can be measured by the number of previous works it makes superfluous to study (quoted by Neugebauer 1957: 145).

The emergence of geometry from ritual analysis is likely to have happened earlier than that of linguistics from the analysis of recitation because of the chronological priority of Baudhāyana: Pāṇini probably belonged to the fourth century B.C., but the Baudhāyana Śulba Sūtra may have been compiled before 500 B.C. (Pingree 1981: 4). The important Baudhāyana Śrauta Sūtra is certainly earlier: 8th or 7th century B.C. (see Staal 1989: 305 and the literature cited there).

The development of scientific geometry is exemplified by the construction mentioned by Seidenberg and van der Waerden, which is found in the Śulbasūtras of Baudhāyana, Āpastamba and Kātyāyana. It exemplifies both strengths and weaknesses of Greek and Vedic mathematics. I mentioned that both traditions lacked simple expressions for numbers; but they share deficiencies that go deeper. Both lacked a symbolic notation such as is used in modern algebra, a discipline Europe imported from the Arabs who developed it from what they had inherited from ancient Mesopotamia and discovered in India and China. The Arabs combined and were accordingly in a position to disentangle the confusing muddle of number references and notations they had inherited from their predecessors: *words* which had been employed by all ancient peoples; Greek *letters*; and Indian *numerals* which ultimately prevailed and then led to the next step which combined them again with letters viz., algebra. The only Greek scientist who used letters in such a manner as *variables* had been Aristotle and that was not in mathematics but in logic.[2]

Instead of modern algebra, the ancient Greeks and Vedic Indians possessed what has been called *geometrical algebra*, an algebra that makes use of geometrical methods and that we can only understand adequately if we desist from reading the expressions of modern algebra into it. The situation is similar to "Newton's equations" which I discussed in the 1995 issue of this *Journal* (23: 76) where I quoted the historian of science C. Truesdell: "It is true that we, today, can easily read them into Newton's words, but we do so by hindsight." We similarly tend to read modern equations into early medieval Indian algebra, which was just beginning to develop a symbolic notation, and are predisposed to do the same with respect to the much earlier phases of Vedo-Greek ritual geometry.

I shall begin with geometrical algebra, following Seidenberg. In the following figures, capital letters refer to points and line segments. I shall use small letters to refer to lengths and express their relationships through the language of modern algebra. In that language, the entire ancient edifice of geometrical constructions is reduced to one line: if the sides of the given rectangle are a and b, we are seeking a square with side c such that:

$$c^2 = a \cdot b \text{ and hence } c = \sqrt{a \cdot b}.$$

(So much for notation; to make it effective, we need to know how to extract a square root.)

In the Indo-Greek geometrical algebra, the construction of a square equal in area to a given rectangle is carried out in two steps. In the first, the rectangle is transformed into a difference of two squares. In the second, this difference is made equal to a square with the help of the Theorem of Pythagoras. Step 1 is illustrated by Figure 4 (from Seidenberg 1983: 99, which follows Thibaut's translation of Baudhāyana's *Śulbasūtra* 1.54).

The given rectangle is ABCD. Now carve out a square AEFD with sides equal to AD. The remainder is EBCF. Divide it into two equal rectangles, EGHF and GBCH. Place one of these on top: DFIJ. Fill the empty space in the corner by the small black square: FHKI. In so doing we have created a larger square: AGKJ. Because of the move of the hatched rectangle from the right to the top, it is easy to see that the irregular area without that small square, viz, AGHFIJA, has the same area as the original rectangle ABCD. The difference between the two

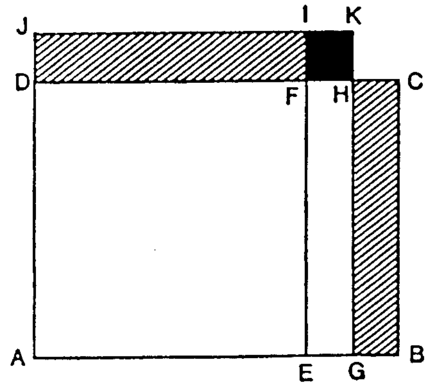


Figure 4. Transformation of a rectangle into a difference of two squares.

squares AGKJ and FHKI is the area of the original rectangle. End of Step 1.

To construct the difference of two squares geometrically (Step 2), use is made of the Theorem of Pythagoras as in Figure 5:

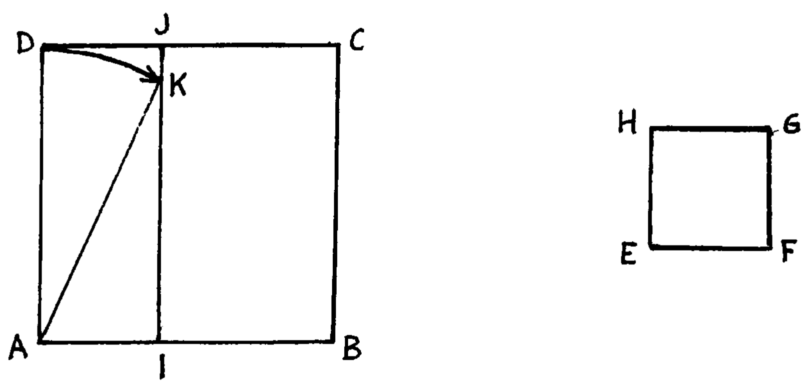


Figure 5. Constructing a square equal to the difference between two squares with the help of the Theorem of Pythagoras.

Since Seidenberg does not provide it, I shall improve Thibaut's translation of Baudhāyana's *Śulbasūtra* 1.51 (in Satyaprakash and Sharma 1968: 55) by following Hayashi 1995: 107 and interpret it (between square brackets) in terms of Figure 5: "One who wishes to deduct one square [EFGH] from another square [ABCD] should cut off a piece from the larger square with the side [AI] of that square one wishes to deduct [i.e., so that AI equals EF]. One should draw the longer side [AD] of the cut-off piece [AIJD] obliquely up to the opposite side [making the arc DK]. One should cut that side [IJ] at the point [K] where it [AD] falls. By the line [IK] which has been cut off, the small square is deducted."

Baudhāyana divided the Theorem of Pythagoras into two complementary parts: proposition 1.51 relates to the difference between two squares; the addition had been formulated in 1.50 (corresponding to Euclid I.47). Note that Step 2 may be incorporated in Step 1 by "drawing" (in Figure 4) KG of the cut-off piece KGEI "obliquely" (*akṣṇayā*).

I have already expressed the problem in the simple language of modern algebra. If we combine Seidenberg with van der Waerden (1983: 11), we may depict its original Vedo-Greek complexity in algebraic terms, as follows. Refer to the base AB of the original rectangle of Figure 4 as a and to its height AD = BC as b. The area of that rectangle is $a \cdot b$. The area of square AEFD with sides equal to AD is b^2. The area of the remainder EBCF is $(a - b) \cdot b$. The area of each of the two equal rectangles EGHF and GBCH into which it is divided is $((a - b) \cdot b)/2$. After placing one of these on top, the area of the small square FHKI which fills the empty space in the corner is $((a - b)/2)^2$ or $(a^2 - 2ab + b^2)/4$. The area of the larger square AGKJ which has been created is $(b + (a - b)/2)^2$ or $((a + b)/2)^2$ or $(a^2 + 2ab + b^2)/4$. With the help of algebra, it is easy to see that the difference between the two squares is $a \cdot b$, i.e., the area of the original rectangle.

Now the deduction of the smaller square from the larger one. Call the side of the second b, that of the first a and that of the difference between the two that has to be found: c. If, in Figure 5, AI = b and AK = AD = a, IK is the side of the square, c, since: $c^2 = a^2 - b^2$ – an algebraic expression of Pythagoras' Theorem.

Seidenberg discovered that the same two steps were taken by Euclid in Book II of his *Elements*. Van der Waerden concludes:

> This is only one of several possible constructions. The fact that Euclid and the *Śulvasūtras* both use one and the same more complicated construction, based on the Theorem of Pythagoras, is a strong argument in favour of a common origin.

The Vedic geometrical rituals emphasized equality of *area* whereas the Greeks emphasized equality of *volume*. Since volume is more complex than area, this suggests that the Vedic geometry might have been the earlier of the two, or a more direct survival of the original. Is Vedic originality in the domain of ritual geometry likely in terms of what is known of Vedic history? Can it be evaluated from the perspective of the Vedas themselves?

III. GEOMETRIES ON THE MOVE

The only two Vedas that concern us are the *Rig*- and *Yajurveda*. The Rigveda was composed during the second millenium B.C. (the bulk of it before 1100 B.C.: Witzel 1992: 614); but it does not mention the Agnicayana and refers to altars only thrice. In each case, the altar seems to be simple and part of the *domestic* ritual, performed within the home and different from the Agnicayana ritual which is a (relatively) "public" ritual. In the one (late) case in which the Rigveda description implies a shape, it is quadrangular (RV 10.114.3; cf. Potdar 1953: 73). In the domestic ritual of a few centuries later, about which more evidence is available, the altar is circular. Though not referred to, it is likely that this circular altar was already known at the time of the Rigveda.

The Yajurveda was composed between 1000 and 800 B.C. and a full third of it is devoted to the Agnicayana. The language is still similar to that of the Rigveda but begins to develop in different directions (see Renou 1956: Ch. I; Witzel 1989). Whatever the difference in language between the two Vedas with their numerous subdivisions, it is likely that there were also social and ethnic differences among its users. Most of the Rigveda was probably composed by members of the semi-nomadic Indo-European pastoralists whose tribes and clans had trickled in across the mountain ranges that separate Central Asia from Iran and the Indian subcontinent; whereas parts of the Yajurveda are more likely to have been composed by indigenous Indians who had become bilingual by adopting the language of these incoming tribes as a second language (cf. Deshpande 1993). Why they adopted this alien language has not been satisfactorily explained; but the fact is not in doubt and not confined to India: it holds true of Indo-European in general (see Mallory 1989: 257–261).

It is clear that nomads do not carry bricks across high mountain passes and are unlikely to construct altars from them. It would stand to reason, then, to assume that the *bricks* of the Agnicayana were a product of Indian soil. This thesis was defended by H. S. Converse in an important article of 1974. Her argument for the indigenous origin of the Agnicayana was not only based upon the hypothesis that the art of baking bricks is unlikely to have been known to nomads, but upon the more important positive fact that it was common in Northwest India in the early second millenium B.C., from the time of Indus Civilization cities such as Harappa. Converse (1974: 83) writes: "The Harappans used millions of kiln-fired bricks as well as countless sun-baked ones." (No bricks were fired in kilns in the urban civilizations of ancient Mesopotamia.)

It seems likely, then, that the *technology* of the Agnicayana, an important part of the ritual, was Indian.[3] This conclusion is strengthened by Converse's further observations that the Yajurveda is familiar with the technique of firing bricks in kilns and makes frequent use of a term for brick – *iṣṭakā* – which does not occur in the Rigveda.

A problem with Converse's hypothesis is that the ancient Iranians referred to bricks as *ištya* which is a cognate of the *iṣṭakā* of the Vedic Indians (Staal 1983, I: 132). Seidenberg (1983: 123) noted that the argument of the invading nomads not having bricks would have to come to terms with this fact. The solution of that puzzle was provided unwittingly by Michael Witzel a decade later: according to him, the Indo-Iranian term for brick is one of several terms which the travelling Indo-Europeans took from the settled peoples of the "Bactro-Margiana archaeological complex" at a much earlier period (during the last centuries of the third millenium B.C.: Witzel 1992: 617).

I have already referred to these Margian-Bactrian or Bactro-Margian peoples which are well known to archaeologists, especially Russian archaeologists, who made numerous excavations at this "Bactrian-Margianian Archaeological Complex" (BMAC). Though the BMAC civilization lasted only a few hundred years, between 2000 and 1500 B.C., much recent information is available on the abundant traces of its influence in the Indo-Iranian borderlands. According to Hiebert (1995: 199), "it is tempting to see the spread of the BMAC to Iran and South Asia as the diffusion of a new type of political-economic and linguistic structure carried by a small group of people and adapted by local populations" (see also Hiebert and Lamberg-Karlovsky 1994: 12).

I shall not try to pursue the archaeological ramifications of these discoveries which are numerous, but provide one illustration. Figure 6 provides the reconstruction by Viktor Sarianidi (1987: 52) of the Togolok-21 temple, dated to about 1800–1500 B.C., a large rectangular complex (130 by 100 meters) with round corner towers and in one corner, on a plot especially reserved for them, *two round brick-faced altars dug into the earth*.

I believe we are now beginning to see the first outline of a demonstration of our hypothesis regarding the geographical area of origin of Vedic and Greek geometry. It is only a beginning, but it points in the right direction, and archaeologists have to take it from there.

One more question arises. Since the techniques of firing bricks was known to the Indus Valley Harappans, were they also linked to this BMAC complex? This question has to a large extent been answered

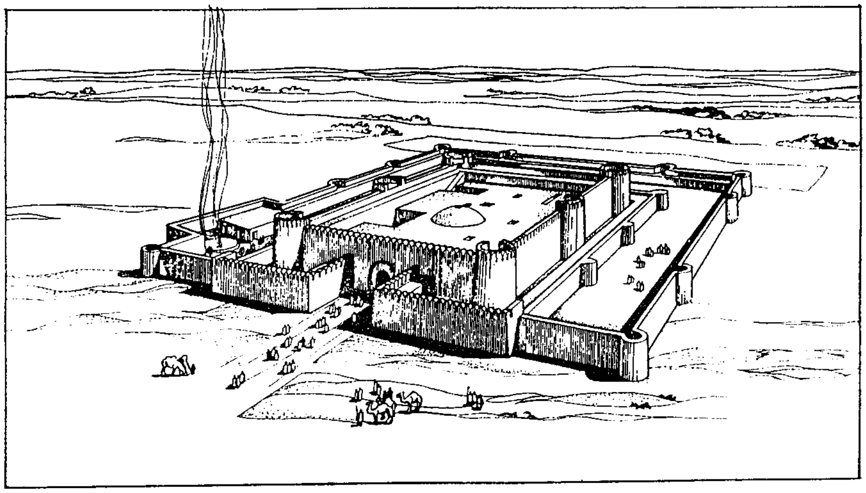


Figure 6. Reconstruction of Togolok-21.

by French archaeologists, especially Jean-François Jarrige, who has established the existence of extensive cultural exchanges between the two areas starting during the end of the third and continuing through the beginnings of the second millenium B.C. (see, e.g., Jarrige 1985 and cf. Parpola 1993: 47–48). Jarrige and his colleagues have also excavated altars in other early Indian cultures which display similarities with Vedic ritual. I do not know and cannot judge the detail of these excavations, but Figure 7 illustrates an example of an altar from Pirak in Baluchistan, dating from roughly the middle of the second millenium B.C. (from Jarrige, Santoni and Enault 1979 II: 9).

This altar exhibits the kind of pattern displayed by the bricks of the smaller so-called Dhiṣṇya Hearths of the Vedic Soma ritual which is part of the Agnicayana. There are six of these hearths, they are sun-, not kiln-fired, each consists of a single layer and the square is of the same size as the basic square of the Agnicayana. The bricks are consecrated with mantras in a specific order. Figure 8 provides two illustrations (from Staal 1983 I: 588–589).

Archaeological similarities or even contacts do not provide adequate grounds for speculating productively about scientific contacts. But there have to be contacts for there to be scientific contacts and some of our facts throw light on the transmission of ritual geometry. As for contacts, it cannot be without significance that the general area of Bactria (sometimes independent but more often ruled by others, including Persians, Tocharians, Indo-Scythians and Arabs) has always

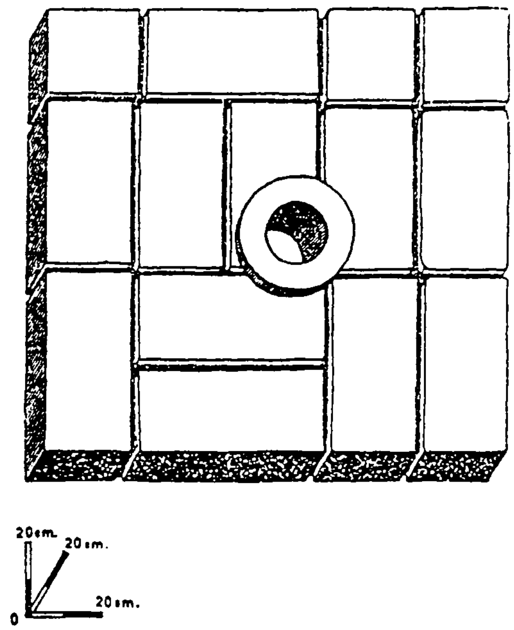


Figure 7. Altar from Pirak, Baluchistan.

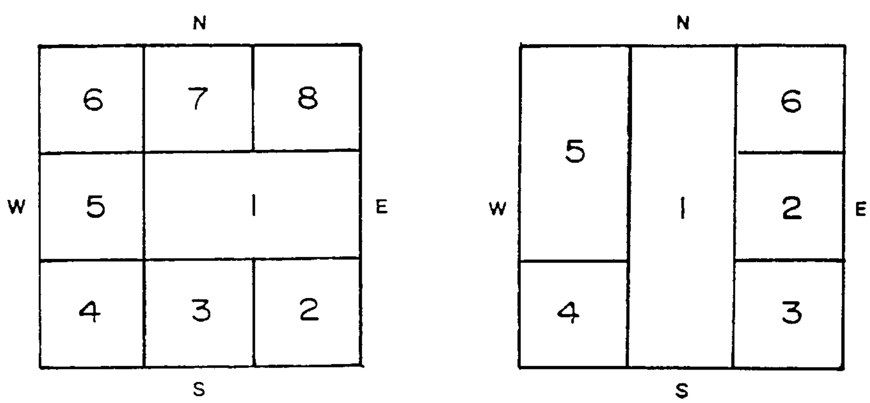


Figure 8. Dhiṣṇya Hearths from the Agnicayana.

been one of the most important junctions on the trade routes between West, East and South Eurasia. Contacts between the Indus and Oxus regions started early and continued through historical times (Jettmar 1986: 145–146; Brough 1965, republished 1996: 276–312). Bactria does not occur among the civilizations compared by van der Waerden (1983: 44–45) or Joseph (1990: 10, 14) but Joseph refers in his account of

Arab mathematics (page 304) to the Kushan empire: “Indian astronomy was transmitted to Central Asia, probably during the first and second centuries AD, when such regions as Khwarizm, Parthia, Sogdiana and Bactria, as well as much of northern India, were part of the great Kushan empire”. The importance of the entire area is unsurprising for it is the heart of Eurasia. But then why is mathematical knowledge not equally distributed along all these routes and throughout the continent? It is true that there are links of ancient Mesopotamia with ancient Greece, India and China, especially in the area of astronomy; but the only really close similarities in mathematics pertain to the Vedo-Greek geometry.

That closeness is explained by the Indo-European background which also illustrates how ritual geometry was transmitted along with language and ritual. Togolok is about equally distant from Athens and Banaras: almost 2,000 miles as the crows flies. Our nomads and semi-nomadic pastoralists did not fly. They had to pass through deserts, circumvent rivers, lakes or seas and cross high passes, often carrying food and other necessities. Horses or camels assisted them on many occasions. They did not carry bricks, notebooks or paper. The transmission of their knowledge was oral for it preceeded the invention of writing and numerals. That ritual geometricians went along on these expeditions and carried not only their language but also their science in their heads is not only likely. There is plenty of evidence that supports it.

The transmission was facilitated in the first place by ritual performances since much of the geometry was *inherent* in the ritual, just as inherent as algebra is in geometry and geometry in algebra as it turned out later (and only more fully in Descartes’ analytical geometry). At least equally important, rituals were performed *en route* as is apparent from a variety of facts that combine to show that Vedic ritual was, at least originally, a *nomadic* ritual. The first fact is that Vedic ritual is performed within a temporary enclosure made of perishable materials, unlike the later Hindu temples and such monuments as remain from the city civilizations of Mesopotamia and Harappa or have been excavated at places like Togolok. The enclosure is used once and it became customary to set it to fire afterwards. All ritual implements (cups, goblets, ladles, pots etc., whose particular shapes continue to be orally transmitted) are similarly prepared for a single performance and afterwards destroyed. Only if bricks are used, they tend to remain, especially in Central Asia; less so in South Asia where they gradually disappear into the soil which is quick to take over in the jungle of a humid and tropical climate (see Staal 1983 I: 188. Plate 16).

That the ritual was originally nomadic is indicated by a variety of other facts. Ritual manuals prescribe that the payment of fees to priests should preferably be made in the form of cows to which some other animals may be added; or the sacrificer may give his daughters in marriage to some of the priests; but payment cannot be made in the form of land. The *Kauṣītaki Brāhmaṇa* of the Rigveda (16.5.1) wonders in a similar spirit why brahmans and kṣatriyas, unlike sedentary vaiśyas, are "transitory, wandering" (*pracyāvuka*), i.e., "do not have a fixed place of residence" (Sreekrishna Sarma 1983: 166–167). Most telling are the *yātsattra* rituals which Heesterman (1962: 34) describes (see also Hillebrandt 1897: 158–159, with references). These were performed by people trekking along a river: each day they threw a wooden peg and where it settled, they stayed on the next day and performed their ritual. Some of their larger ritual receptacles were moved on wheels.

Our investigation started from the fact that the geometry of the Vedic Indians and ancient Greeks was based on rituals which required that changes be made in the size and shape of altars and bricks. From that ritual background a more purely theoretical form of geometry developed just as (in India) linguistics arose from the practice of recitation. Prior to attaining that more universal and scientific level, the geometry was inherent in rituals that wandering nomads or semi-nomads performed on their journeys. Ritual and ritual geometry went the same ways as the Indo-Aryan or Proto-Indoaryan languages: the majority of speakers went from Central to South Asia; but a tiny group reached the shores of the Eastern Mediterranean and left enough elements of ritual geometry behind for a kindred form of geometry to develop and flourish also there.[4]

NOTES

[1] "Bref, c'est le Yajurveda qui demeure la base du culte et qui sans doute a déterminé toute l'évolution du védisme littéraire" qui "forme un tout cohérent, ou chacque partie s'est organisée en liaison systématique avec les autres": Renou 1947: 9, 209–210.

[2] Lukasiewicz (1957: 7–8) expressed surprise that no philosopher or philologist had paid attention to this discovery except the Oxford editor of most of Aristotle's works, Sir David Ross. Otto Neugebauer was more explicit: "The origin of algebra is totally independent of an algebraic notation in one of the most famous philosophical works of antiquity" (1957: 225).

[3] Kashikar (1979) has argued against this conclusion (as formulated in Staal 1978) and against any "so-called pre-Vedic element"; but see Staal (1982: 42–47). That the Harappans possessed a highly developed system of weights and measures was shown by Chattopadhyaya 1986; but it is not reflected in the Vedas and nothing is known about Harappan mathematics (Hayashi 1994: 126).

[4] As in the case of germs, paucity of carriers or wandering geniuses does not

diminish the force of these conclusions. The further we go back in time, the smaller the number of people who contributed decisively to the development of human skills and knowledge. Greek mathematics itself consists of the fragments of writings of about 10 or 20 persons scattered over a period of 600 years (Neugebauer 1957: 190). The greatest, perhaps, of all human inventions or discoveries, the taming of fire, may have been the single most important event in 10,000 years of human evolution.

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